ENTANGLEMENT IN LOCAL SYSTEMS

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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In this study, we first discuss entanglement measures and we introduce a way to construct generic entangled states of an n-level quantum system.

Then we discuss entanglement as a local object. Particularly we use a spin qutrit, and investigate whether an entangled spin qutrit obeys entanglement criteria or not. While doing this, we also discuss, which criteria of entanglement are essential and which of them are not. We show the relation between quantum fluctuations and entanglement.

Lastly, we discuss Bell type inequalities and we show violation of a Bell type condition by a single particle entangled state.

Keywords: Quantum entanglement, entanglement measures, quantum information, foundations of quantum mechanics, entanglement and quantum nonlocality, Bell type inequalities.

ÖZET

YEREL SISTEMLERDE DOLANİKLİK

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Bu çalışmada, öncelikle dolanıklığın ölçüsü tartışılmış ve n-seviye bir kuvantum sisteminin genel dolanıklık hallerini kurmak için bir yöntem tanıtılmıştır.

Sonra dolanıklık yerel bir nesne olarak tartışılmıştır. Özellikle bir spin üç seviyeli sistemi kullanılmış ve dolanık bir spin üç seviyeli sistemin dolanıklık kriterlerine uyup uymadığı araştırılmıştır. Ayrıca bu yapılırken, dolanıklığın hangi kriterlerinin gerekli olup hangilerinin gerekli olmadığı tartışılmıştır. Kuvantum dalgalanmaları ile dolanıklık arasındaki ilişki gösterilmiştir.

Son olarak, Bell tipi eşitsizlikleri tartışılmış ve Bell tipi bir şartın, tek parçacık dolanık hali tarafından ihlal edildiği gösterilmiştir.

Anahtar sözcükler: Kuvantum dolanıklık, dolanıklık öçüleri, kuvantum enformasyon, kuvantum mekaniğinin temelleri, dolanıklık ve kuvantum yerelsizliği, Bell tipi esitşizlikler.

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This thesis is a product of almost three years that I have spent with Prof. Shumovsky. Within these three years, he was an “advisor” to me not only with his knowledge but also with his attitude towards life.

The working group that I have been into for these years, had accomplished very good jobs, with Prof. Shumovsky’s unending enthusiasm, Prof. Klyachko’s creativity, M. Ali Can’s wisdom, and Özgür Çakir’s ingenuity. I wish I could have spent more time with them.

In life, most of us eat, drink, sleep while dealing with daily troubles, just a few of us really live. Prof. Shumovsky lived a very beautiful life with his humor, knowledge, good manners, and joy of life. I will never forget this person who, by frequently advising me to “calm down”, had let me be able to pause sometimes in life and look around. It is a duty for me to express my gratitude to those people who made my life more meaningful, to my friends and to my family for their support and trust.

I also thank to the jury members Prof. Erçelebi, Prof. Oktel, Prof. Salzner, and Prof. Gündüç, who spent their time to evaluate my work.

With my wish of being worthy, this study is dedicated to the precious memory of Prof. Shumovsky.

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Chapter 1

Introduction

Quantum entanglement is a phenomenon which has no classical analog. Its practical importance could not be understood for a long time. Recent developments of quantum information technologies have led to a number of successful and promising realizations of protocols, based on the use of quantum entanglement. For example, quantum key distribution has recently become an industrial product [1].

Quantum information can be defined as a physical information held in the *state* of the quantum system. Quantum systems are described by quantum states, which are linear combinations (superposition states) of the eigenstates of the observables. If a measurement is done, the state will be randomly collapsed onto one of the eigentstates in the superposition . Superposition of multipartite states is connected with notion of entanglement. Entanglement arises due to nonclassical correlations in a quantum system. Once subsystems of a composite quantum system are interacted, their states may be superposed with one another, the systems may be "entangled".

Development in quantum information science caused a great burst of activity in the investigation of quantum entanglement. During the last decade, the applications of entanglement were discussed by a number of groups all around the world. With the aid of quantum information, we can perform certain tasks, that are not possible classically. For example quantum computers (they can perform some tasks which are difficult or impossible for classical computers) [2]; quantum cryptography (unconditionally secure transmission of information) [3]; dense coding [4], and teleportation [5] became possible with the use of quantum information.

Entanglement is widely considered as a property of nonlocal systems, this is also useful for communication purposes, where individual parties of the system are well separated. We proposed [6, 7] that, entanglement can also be considered as a property of local systems (as well). In some local systems, intrinsic degrees of freedom of a single particle can be entangled. We can easily see the entanglement of such a particle if it decays into two separated, entangled particles (e.g. biphoton). Our concern is, can we observe entanglement before the decay? In fact,we can observe violation of classical realism by such a state.

To make separate measurements on different degrees of freedom of a single particle is a hard task with the existing experimental techniques. However we can argue some principles of entanglement, which of them are essential and which of them are not. And we can check whether our proposition obeys to essential properties of entanglement or not.

Discussion of entanglement as a local object may not be useful for practical purposes (especially for communication purposes) for now, however it is very important for the understanding the physics behind entanglement. It may be useful in the future, especially for computation purposes.

While describing entanglement physically, we should also give a quantative description of it. A technique to quantify entanglement is proposed by our group [8, 9], this technique is based on specifying the quantum system by accessible observables, and it relates quantum fluctuations with entanglement.

The thesis is organized as follows:

In the second chapter; firstly we discuss some entanglement measures, and we introduce variance as a measure of entanglement. Then, an algebraic way to construct generic entangled states of qunits based on the polar decomposition of the su(2) algebra is discussed. In particular, we show that these states can be defined as eigenstates of certain Hermitian operators.

In the third chapter; first we discuss the physical properties of entanglement. We discuss the entanglement of SU(2) qutrit and its correspondence with two-qubits. Then we show the relation between the quantum fluctuations and entanglement. Lastly we propose some physical systems to realize single particle entanglement.

In the fourth chapter, we discuss the Bell's inequalities. Firstly, we recall Bell's original inequality and CHSH (Clauser, Horne, Shimony, and Holt) inequality. Then we introduce our "pentagram inequality", and give violation of this inequality by a single qutrit state.

In appendices, we give deeper explanations of the notions that we used in our text.

Chapter 2

Generic Entanglement

In this chapter we first summarize some techniques to measure entanglement and then introduce Generic Entangled States.

2.1 Entanglement measures

Entanglement was first introduced to quantum mechanics by Einstein to show the inconsistency in the statistical nature of the theory [10]. In the famous paper "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" Einstein, Podolsky, and Rosen discussed the uncertainty principle. They considered two separate systems I and II, which interacted for a time but have no interaction after a while. By performing simultaneous measurements to each subsystem, they concluded that it is possible to predict physical quantities described by noncommuting operators. However according to quantum mechanics, if two operators are noncommuting, correlation type measurements are not possible. Those discussions led to remarkable proposal of the states that manifest *quantum nonlocality* and to an attempt to adjust these *spooky states* with the "classical common sense" by means of the *hidden variable* modification of quantum mechanics. Nonlocality means that, if a measurement is done on one part of the separated quantum system, this measurement can also influence other parts, due to nonclassical correlations between them. Hidden variable theories are based on two joint assumptions; existence of reality before observation (realism), and locality. Bell was not convinced with the hidden variables in quantum theory. He introduced his famous inequalities to show nonexistence of hidden classical variables in quantum mechanics [11].

Entanglement can be associated with nonclassical correlations in a quantum system. Detection and quantification of entanglement is important for both theoretical and practical aspects. Although Bell's inequalities were introduced to show nonexistence of hidden variables in quantum mechanics, they also served to detect the entanglement in a given quantum system.

Separability condition is another criterion for entanglement. Nonseperability can be related with nonlocality principle. For pure states by decomposing the states into Schmidt form [12] one can easily check the separability of the state. For mixed states, separability condition becomes harder, for them entanglement witnesses can be used as a criterion [13]. Although the separability criterion works well for bipartite systems, it has no meaning for single particle system. Also it does not work well for some three partite states (GHZ and W-states).

To give a quantitative measure of entanglement is harder then to give a qualitative test. For this aim we need operations that can be applied to quantum system and that can create or increase only classical correlations (not quantum correlations). So entanglement cannot be created via such operations. If a number assigned to the state is not increasing under such operations it can be considered as an entanglement measure. Any scalar valued function that satisfies this criterion is called an *entanglement monotone*. The operations mentioned above are called *local operations assisted by classical communications* (*LOCC*)[14]. LOCC imply general local operations, and also allow for classical correlations between them. There is another class of operations called *stochastic local operations assisted by classical communications* (*SLOCC*). SLOCC is more useful especially for multipartite states. [15, 16, 17].

SLOCC can be described as classification of entanglement, which is a coarse-grained classification of entanglement under LOCC [18].

Some important properties of entanglement monotone can be summarized as; it should be invariant under local unitary transformations, should be zero for a separable state, should take maximum value for Bell states, and should give asymptotic conversion rate from an arbitrary state to a standard Bell state.

Below some entanglement measures are summarized:

2.1.1 Information Entropy

Claude Shannon established some core results in classical information theory [19]. Shannon entropy quantifies the amount of uncertainty in the system (in other sense lack of knowledge). Let X be a discrete random variable taking a finite number of possible values $x_1, x_2, \cdots, x_n$ with probabilities $p_1, p_2, \cdots, p_n$ ($\sum_{i=1}^{n} p_i = 1$). Shannon entropy can be formulated as:

$$H(p_1, p_2, ..., p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i. \tag{2. 1}$$

Entropy becomes maximum if we do not have any information about the outcomes of the measurement, i.e. they are all equally likely. For bipartite case binary entropy takes the form

$$H(p) = -p \log_2 p - (1-p) \log_2 (1-p) \tag{2. 2}$$

where p and $1-p$ are the probabilities of the two outcomes.

In the case of quantum information, random variables become density matrices. *Von Neumann entropy* is the quantum counterpart of the classical Shannon entropy [20]:

$$S(\rho) = -Tr(\rho \log_2 \rho) \tag{2. 3}$$

In terms of the spectrum α_i of the density matrix ρ we can convert equation to the following:

$$S(\rho) = -\Sigma_i \alpha_i \log_2 \alpha_i \tag{2. 4}$$

It is invariant under unitary transformations (depends only on eigenvalues).

Von Neumann entropy of a pure state is zero, whether it is entangled or not. For a completely mixed state $\rho = I/n$ in $n - dimensional$ system it takes maximum possible value $S(\rho) = \log_2 n$. But if we take the von Neumann entropy of the reduced density matrices we will get an entanglement monotone called *entanglement of formation*[21]:

$$E(\psi) = -Tr(\rho_r \log \rho_r) \tag{2. 5}$$

for a pure state and

$$E(\rho) = min \Sigma p_i E(\psi_i) \tag{2. 6}$$

for a mixed state. It is not important which subsystem we are using. Even if their dimensions are different they have the same nonvanishing eigenvalues, and this is the part that is invariant under unitary transformations. Entanglement of formation takes the value zero for separable states. Let's calculate it for Bell state:

$$\rho = |\Psi^+_{Bell}\rangle\langle\Psi^+_{Bell}| = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2. 7}$$

Partial traces become

$$\rho_1 = \rho_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{2. 8}$$

and entanglement of formation takes the maximum value (one) for the Bell state. It gives us the degree of quantum correlation in the system.

For mixed states, to find an analytic solution of entanglement of formation is very hard. But for two qubits, an analytic expression is introduced by Wootters [22, 23].

2.1.2 Concurrence

Wootters's starting point was a useful fact about pure states of two qubits. If we define the orthonormal basis of the four dimensional Hilbert space of two qubits in terms of Bell states with different phases (*magic basis*):

$$\begin{aligned}
|e_1\rangle &= \tfrac{1}{2}(|00\rangle + |11\rangle) \\
|e_2\rangle &= \tfrac{1}{2}i(|00\rangle - |11\rangle) \\
|e_3\rangle &= \tfrac{1}{2}i(|01\rangle + |10\rangle) \\
|e_4\rangle &= \tfrac{1}{2}(|01\rangle - |10\rangle)
\end{aligned} \tag{2. 9}$$

an entanglement monotone called *concurrence* of a pure state $|\psi\rangle = \sum_i \alpha_i |e_i\rangle$ can be defined very simply [22]

$$C(\psi) = |\sum_i \alpha_i^2|. \tag{2. 10}$$

There is a strong relationship between concurrence and entanglement of formation:

$$E(\psi) = H(\frac{1}{2}(1 + \sqrt{1 - C^2})) \tag{2. 11}$$

Hence, once we find concurrence we can easily calculate entanglement of formation for a pair of qubits. Concurrence can also be considered as a measure alone. An analytic expression of concurrence for mixed states is also present [23]. For this aim we should calculate spin-flip transformation of the density matrix

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{2. 12}$$

where the ρ and complex conjugation ρ^* is taken in the standard basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. In fact spin-flip transformation is just complex conjugation in the magic basis. Concurrence for any bipartite two level system can be formulated as following:

$$C(\rho) = max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{2. 13}$$

where λ_i's are square roots of the spectrum of the matrix $\rho\tilde{\rho}$, in decreasing order.

We can write a general two-qubit state as following

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \tag{2. 14}$$

and concurrence for such a state becomes

$$C(\psi) = 2|ad - bc|. \tag{2. 15}$$

There also exists an analytic measure of entanglement for three qubits. Three-qubit states may manifest entanglement of two different types. Namely, entanglement caused by correlations of all three qubits and entanglement due to correlation between pair of parts [15]. The three-qubit entanglement is measured by means of 3-tangle [17, 24], which for the general state

$$|\psi\rangle = \sum_{k,\ell,m=0}^{1} \psi_{k\ell m}|k\ell m\rangle, \qquad \sum_{k,\ell,m=0}^{1} |\psi_{k\ell m}|^2 = 1,$$

has the form

$$\begin{aligned} \tau(\psi) = 4|\psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2 \\ -2(\psi_{000}\psi_{001}\psi_{110}\psi_{111} + \psi_{000}\psi_{010}\psi_{101}\psi_{111} + \psi_{000}\psi_{100}\psi_{011}\psi_{111} \\ +\psi_{001}\psi_{010}\psi_{101}\psi_{110} + \psi_{001}\psi_{100}\psi_{011}\psi_{110} + \psi_{010}\psi_{100}\psi_{011}\psi_{101}) \\ +4(\psi_{000}\psi_{011}\psi_{101}\psi_{110} + \psi_{001}\psi_{010}\psi_{100}\psi_{111})| \end{aligned} \qquad (2.\ 16)$$

According to classification by Miyake [17], the following three states

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \qquad (2.\ 17)$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle), \qquad (2.\ 18)$$

$$|Bi\rangle = \frac{1}{\sqrt{2}}(|011\rangle + |101\rangle) \qquad (2.\ 19)$$

are generic for the three SLOCC-nonequivalent classes in the eight-dimensional Hilbert space. The *non-separable* states from the GHZ class manifest only three-partite entanglement (3-tangle (2. 16) has nonzero values for the states from this class), while any pair of qubits is unentangled. The latter can be checked by reduction of the three-qubit density matrix $\rho_{GHZ} = |GHZ\rangle\langle GHZ|$ to the two-qubit mixed state $\rho'_{GHZ} = \mathrm{Tr}_{single}\rho_{GHZ}$, where Tr_{single} denotes trace over one of the parts, with the subsequent calculation of the concurrence, which in this case always has zero value.

In turn, the *non-separable* states from the W class always have zero 3-tangle and hence do not manifest three-partite entanglement, and any bipartite reduced state with the density matrix $\rho'_W = \mathrm{Tr}(|W\rangle\langle W|)$ has nonzero concurrence and therefore shows bipartite entanglement.

Finally, the *separable* states from the Bi class are similar, in a sense, to the W states. Namely, they always have zero 3-tangle while manifest bipartite entanglement (for two given qubits only).

Thus, the nonseparability (separability) of the three-qubit states does not indicate identically the presence (absence) of entanglement and its type in contrast to the bipartite systems.

In the case of three-qubits, Von Neumann entropy does not work. Consider as an example the GHZ-type state of the form $|\Psi\rangle = x|000\rangle + y|111\rangle$, $x^2 + y^2 = 1$. It is seen that 3-tangle (2. 16) $\tau(\Psi) = 4x^2y^2 = 4x^2(1-x^2)$, so that the state is entangled (in the three-part sector) for all $x \in (0,1)$. The reduced two-qubit density matrix for any pair of qubits has the form

$$\rho_R = x^2|00\rangle\langle 00| + (1-x^2)|11\rangle\langle 11|$$

with the corresponding von Neumann entropy

$$H(\rho_R) = -x^2 \log x^2 - (1-x^2)\log(1-x^2).$$

Subsequent reduction of ρ_R to the single-qubit state

$$\rho_{RR} = x^2|0\rangle\langle 0| + (1-x^2)|1\rangle\langle 1|$$

obviously leads to the same von Neumann entropy $H(\rho_R)$ as ρ_R, although there is no two-qubit entanglement in the state. Similar behavior, showing unfitness of the reduced state entropy as a general measure of entanglement, is manifested by the W and Bi states of three qubits as well.

There is a need of a measure of entanglement based on physical manifestations of entanglement in the process of measurement of quantum observables, which is related with total amount of quantum correlations in the given system.

2.1.3 Variance as a Measure of Entanglement

There is a certain interdependence between quantum correlations peculiar to entangled states and quantum uncertainties (fluctuations) of *local* observables

[8, 9, 25]. Consider as an illustrative example the measurement of spin projection onto the quantization axis in the two-qubit states $|\psi_{00}\rangle = |00\rangle$ and $|\psi_{CE}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. For the correlation functions and variances (uncertainties), we get

$$\begin{array}{ll} \langle\psi_{00}|\sigma_z^A;\sigma_z^B|\psi_{00}\rangle = 0, & V(\sigma_z^{A,B};\psi_{00}) = 0, \\ \langle\psi_{CE}|\sigma_z^A;\sigma_z^B|\psi_{CE}\rangle = 1, & V(\sigma_z^{A,B};\psi_{CE}) = 1. \end{array}$$

Here σ_z^A, σ_z^B denote the z-component of Pauli spin operator,

$$\langle\psi|\sigma^A;\sigma^B|\psi\rangle = \langle\psi|\sigma^A\sigma^B|\psi\rangle - \langle\psi|\sigma^A|\psi\rangle\langle\psi|\sigma^B|\psi\rangle$$

is the correlation function of local measurements, and

$$V(\sigma;\psi) = \langle\psi|\sigma^2|\psi\rangle - \langle\psi|\sigma|\psi\rangle^2$$

is the variance of the observable σ in the state ψ. Thus, the correlation functions and variances have similar behavior for unentangled states like ψ_{00} and entangled states like ψ_{CE}.

The natural question now is how many physical observables should be measured in order to conclude that a given state of a certain system is entangled [26]? This question has extremely high importance for understanding of physical essence of entanglement and its quantification. Besides that, this question has a quite practical meaning in connection with test of sources of entangled states [27].

In a recent approach [8, 28, 29] (for recent review, see Ref. [30]), it has been proposed to begin the analysis of entanglement with the choice of independent *basic observables* that can be associated with the orthogonal basis of a certain Lie algebra $\mathcal{L}$. The corresponding Lie group $G = \exp(i\mathcal{L})$ defines the *dynamic symmetry* of the physical system under consideration.

It should be emphasized that the idea to specify a quantum system by accessible observables is known for a long time (e.g., see [31]). Unfortunately, this principle idea is often set aside. As we show below, this principle plays extremely important role in description of quantum entanglement.

Within the approach of Refs. [8, 28, 29], the *complete entanglement* is interpreted as *manifestation of quantum uncertainties of all basic observables at their extreme*. By complete entanglement we mean here the maximal entanglement that can be achieved by pure states.

Note that, for a given quantum system, it is enough to know the completely entangled states because all other entangled states can be generated from those states through the use of SLOCC [15, 16].

We will discuss the characteristic features of this approach, using a single *qutrit* (ternary quantum state) as an illustrative example of some considerable interest.

Qutrit is usually associated with ternary unit of quantum information [32]. Instructiveness of this example lies in the *relativity of entanglement* with respect to the choice of dynamic symmetry G of ternary quantum physical system. Namely, one can choose either $\mathrm{G} = \mathrm{SU}(3)$ [33] or $\mathrm{G}' = \mathrm{SU}(2)$. Just the latter case of a single spin-1 system may manifest entanglement without division of the system into separated parts [6, 28, 34].

As mentioned, specifying a given quantum system, we should first choose the accessible independent physical observables associated with dynamic symmetry of the system.

For example, in the case of a qubit (spin 1/2) system, dynamic symmetry is given by the group SU(2). The orthogonal basis of the corresponding Lie algebra $su(2)$ consists of three spin operators (Pauli matrices). Thus, a two-qubit system is characterized by the dynamic symmetry $\mathrm{G} = \mathrm{SU}(2) \times \mathrm{SU}(2)$, which corresponds to the six basic observables (three Pauli matrices for each part). For the two-qubit pure state, the number of necessary measurements, providing information about entanglement carried by this state, is reduced to three [26] because of the local character of the measure of entanglement (concurrence) in this case [35].

To illustrate special importance of the specification of quantum system by basic observables, consider a qutrit associated with a state

$$|\psi\rangle = \sum_{s=-1}^{1} \psi_s |s\rangle, \quad \sum_{s=-1}^{1} |\psi_s|^2 = 1 \tag{2. 20}$$

in the three-dimensional Hilbert space $\mathcal{H}_3$. As mentioned, there are at least two qualitatively different physical systems, whose states are qutrits. Namely, one possible realization corresponds to the general symmetry G = SU(3) of the system, which implies eight basic observables (Gell-Mann matrices) [33]

$$\begin{gathered}
\mathcal{O}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{O}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{O}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{O}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{O}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \mathcal{O}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\mathcal{O}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathcal{O}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{gathered} \tag{2. 21}$$

Hereafter we call the corresponding system the *true qutrit* system.

Another realization assumes reduced symmetry G′ = SU(2) of the physical system, which requires only three basic observables (spin-1 operators) [6]

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2. 22}$$

We call this case the *spin-qutrit* system.

As will be discussed in the next chapter, qutrit (2. 20) may manifest entanglement in the case of single spin-qutrit system, while single true qutrit can never be entangled.

Let us briefly discuss the physical definition of entanglement of Refs. [8, 28, 29].

For a given state ψ of a system with basic observables X_i, we can measure the expectation values $\langle\psi|X_i|\psi\rangle$ and variances (uncertainties)

$$V(X_i;\psi) = \langle\psi|X_i^2|\psi\rangle - \langle\psi|X_i|\psi\rangle^2. \tag{2. 23}$$

It is interesting that Wigner and Yanase [36] have associated the variance $V(X_i;\psi)$ (2. 23) with the amount of *quantum information* about the state ψ that can be extracted from *macroscopic measurement* of the observable X_i in this state (see Refs. [37] for further discussion of Wigner-Yanase quantum "skew information").

Following [8, 28, 29], *total variance* can be introduced as

$$V(\psi) = \sum_i V(X_i;\psi) \tag{2. 24}$$

calculated for all basic observables and all parts of the system (in the case of multipartite systems). By definition, this quantity (2. 24) is an invariant, which is independent of the choice of basis of the Lie algebra $\mathcal{L}$ of observables.

This quantity (2. 24) can also be interpreted as the total amount of Wigner-Yanase information peculiar to the state ψ.

It was proposed in Refs. [8, 28, 29] that, *complete entangled states* ψ_{CE} of an arbitrary system can be defined in terms of maximum of total variance:

$$V(\psi_{CE}) = \max_{\psi\in\mathcal{H}} V(\psi). \tag{2. 25}$$

This definition has a simple physical meaning. It associates complete entanglement with the maximal amount of quantum uncertainty in a given system. Validity of this definition in some known cases of completely entangled states of multipartite systems has been shown in a number of papers (see Ref. [30] for references).

It is seen that Eq. (2. 25) represents a certain variational principle, similar in a sense to the maximal entropy principle in statistical physics, which is used to define equilibrium states.

At first glance, Eq. (2. 25) defines only completely entangled states ψ_{CE}. In fact, it can be used to specify all entangled pure states of the system as well. The point is that all entangled states of a given system are equivalent to SLOCC [15, 16]. Note that SLOCC are represented by operators from the complexified dynamic symmetry group [16]

$$\widehat{SLOCC} \equiv g^c \in G^c = \exp(\mathcal{L} \otimes C).$$

Thus, for the entangled states ψ_E we get

$$|\psi_E\rangle = g^c|\Psi_{CE}\rangle. \tag{2. 26}$$

Note that in the case of compact Lie algebra (like SU(N)), the quadratic form

$$\sum_i X_i^2 = C_{\mathcal{H}}$$

is a scalar (Casimir operator). Then Eq. (2. 24) takes the form

$$V(\psi) = C_{\mathcal{H}} - \sum_i \langle\psi|X_i|\psi\rangle^2. \tag{2. 27}$$

It is easily seen that the maximum of the total variance (2. 27) is provided by the condition

$$\forall i \quad \langle\psi_{CE}|X_i|\psi_{CE}\rangle = 0. \tag{2. 28}$$

This condition represents a set of algebraic equations for the complex coefficients of the wave function $|\psi\rangle$, which enables us to fairly simplify the analysis of entanglement. Validity of this condition (2. 28) for completely entangled qubit-states in quite general settings has been checked in Ref. [9]. Because the condition (2. 28) deals directly with measurement of physical observables, it has been proposed in Ref. [9] to use the condition as an *operational definition* of complete entanglement.

Amount of entanglement carried by entangled states (2. 26) can also be measured by means of total variance as follows [38]

$$\mu(\psi) = \sqrt{\frac{V(\psi) - V_{\min}}{V_{\max} - V_{\min}}}. \tag{2. 29}$$

Here $V_{\max}$ and $V_{\min}$ denote the total variance for completely entangled and unentangled states, respectively. This measure coincides with the *concurrence* for pure states of an arbitrary bipartite system. It can also be applied beyond bipartite systems. For unentangled states, $\mu(\psi) = 0$, while for entangled states it lies in $(0, 1]$, so that $\mu(\psi_{CE}) = 1$.

For *n-partite* states $(n > 2)$ general discussion of measure $\mu(\psi)$ can be found at [39]. Here we will discuss three qubits and as an example, following states will be considered:

$$\begin{array}{c} |GHZ\rangle = x|000\rangle + \sqrt{1-x^2}|111\rangle, \\ |W\rangle = x|011\rangle + \sqrt{\frac{1-x^2}{2}}(|101\rangle + |110\rangle), x \in [0,1]. \end{array} \tag{2. 30}$$

For these states we get $\tau(GHZ) = \mu^2(GHZ) = 4x^2(1-x^2)$, and $\tau(W) = 0$ whereas $\mu(W) = \sqrt{(2-6x^4+4x^2)/3}$. For GHZ state, our result is in agreement with 3-tangle measure, but for W state result is quite different. To discuss this lets consider the case $x = 1/\sqrt{3}$. As we discuss earlier, W state contains two-qubit entanglement, so $\mu(W)$ measure contains pairwise correlations, specifically $V(W) = V_{min} + Cov(W)$. Where

$$Cov(W) = \sum_{\alpha=x,y,z} \sum_{i \neq i'} (\langle W|\sigma^i_\alpha \sigma^{i'}_\alpha|W\rangle - \langle W|\sigma^i_\alpha|W\rangle\langle W|\sigma^{i'}_\alpha|W\rangle). \tag{2. 31}$$

We have restricted our consideration by pure states. So far, the measure of mixed-state entanglement is known only for two qubits [23]. The principle difficulty here is that the total variance of mixed states contains contributions of both quantum and classical (statistical) uncertainties. The problem of detachment of the two principally different contributions deserves special discussion. The ideas related to the Wigner-Yanase quantum information [36, 37] may be useful here.

Till now we discussed the quantification of entanglement. Now a regular way will be presented to construct *generic entangled states* of a system consisting of an arbitrary number of local parts with different dimension (qubits, qutrits, etc).

2.2 Generic Entangled States

Our definition of *generic* entangled states coincides with that of Ref. [17, 40]. This assumes that they are completely entangled and have simple structure like Bell and GHZ states of two and three qubits respectively.

We introduced an algebraic way to construct generic entangled states of qunits based on the polar decomposition of the $su(2)$ algebra [41]. By *qunit* we mean here an n-level quantum system, specifying by the observables, forming basis of the $su(n)$ algebra or of its complexification. As we exemplified before, observables for a qubit are specified by Pauli operators, forming an infinitesimal representation of the $s\ell(2, C)$ algebra, which is known to be the complexification of the $su(2)$ algebra, and observables for a qutrit form a Hermitian basis of the $su(3)$ algebra [33]. And so on.

Generic entangled states in multi-qunit systems can be constructed as the $su(2)$ phase states of dimension n. The basis of completely entangled states in the corresponding Hilbert space can be constructed from generic entangled states by means of local cyclic permutation operator. This approach also allows us to specify Hamiltonians, whose eigenstates are the generic entangled states.

2.2.1 SU(2) Phase States

A system of N qunits is defined in the Hilbert space

$$H_{N,n} = \bigotimes_{i=1}^{N} H_n, \quad \dim H_n = n.$$

The basic observables $\mathcal{O}_j$ are associated with the basis of the Lie algebra

$$\mathcal{L}_{N,n} = \bigoplus_{i=1}^{N} su(n)$$

or its complexification. Homogeneous states of N qunits can be written as following

$$|\ell; N\rangle = \bigotimes_{j=1}^{N} |\ell\rangle_j. \tag{2. 32}$$

Using the homogeneous states (2. 32), we can construct an n-dimensional representation of the $su(2)$ algebra of the form

$$\begin{aligned} J_+ &= \lambda_0|0;N\rangle\langle 1;N| + \cdots + \lambda_{n-2}|n-2,N\rangle\langle n-1;N|, \\ J_- &= \lambda_0|1;N\rangle\langle 0;N| + \cdots + \lambda_{n-2}|n-1;N\rangle\langle n-2;N|, \\ J_z &= \frac{n-1}{2}|0;N\rangle\langle 0;N| + \cdots + \frac{1-n}{2}|n-1;N\rangle\langle n-1;N|, \end{aligned} \tag{2. 33}$$

such that

$$[J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm.$$

Thus,

$$\lambda_0^2 = n-1, \quad \lambda_1^2 - \lambda_0^2 = n-2, \quad \cdots, \quad \lambda_{n-2}^2 - \lambda_{n-3}^2 = 2-n, \quad \lambda_{n-2}^2 = n-1.$$

Following Refs. [42, 43], consider the polar decomposition of the $su(2)$ algebra (2. 32)

$$J_+ = J_r E, \quad J_- = E^+ J_r, \quad EE^+ = \mathbf{1}.$$

Here the "radial" operator $J_r = (J_+ J_-)^{1/2}$ is diagonal, while the unitary operator E describes the "exponential of the $su(2)$ phase". It is seen that the operator E has the form

$$\begin{aligned} E &= |0;N\rangle\langle 1;N| + |1;N\rangle\langle 2;N| + \cdots + |n-2;N\rangle\langle n-1;N| + e^{i\varphi}|n-1;N\rangle\langle 0:N| \\ &= \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & & . \\ 0 & 0 & 0 & ... & 1 \\ e^{i\varphi} & 0 & 0 & ... & 0 \end{pmatrix}. \end{aligned} \tag{2. 34}$$

In other words, operator (2. 34) provides cyclic permutations of homogeneous states (2. 32). Here φ denotes an arbitrary "reference phase", which can be putted $\varphi = 0$ for simplicity.

To find eingenstates of the phase operator, consider a linear superposition of homogeneous states (2. 32)

$$|\psi_{N,n}\rangle = \sum_{\ell=0}^{n-1} a_\ell |\ell;N\rangle, \quad \sum_{\ell=0}^{n-1} |a_\ell|^2 = 1, \tag{2. 35}$$

If we operate the state

$$E|\psi_{N,n}\rangle = \alpha|\psi_{N,n}\rangle = \alpha(\frac{a_1}{\alpha}|0;N\rangle + \frac{a_2}{\alpha}|1;N\rangle + ... + \frac{a_0}{\alpha}|n-1;N\rangle),$$

by equating coefficients

$$\begin{aligned} a_1 &= \alpha a_0, \\ a_2 &= \alpha^2 a_0, \\ &... \\ a_{n-1} &= \alpha^{n-1} a_0 = a_0/\alpha \end{aligned} \tag{2. 36}$$

we get $\alpha_k = e^{2i\pi k/n}$, where $k = 0, 1, ..., n-1$

$$E|\psi^k_{N,n}\rangle = e^{2i\pi k/n}|\psi^k_{N,n}\rangle = e^{i\phi_k}|\psi^k_{N,n}\rangle.$$

By normalization

$$|a_0|^2(1 + |\alpha_k|^2 + ... + |\alpha_k|^{2n+2}) = |a_0|^2 n = 1 \tag{2. 37}$$

we get the N-qunit $su(2)$ phase states of the form

$$|\psi^{(k)}_{(n,N)}\rangle = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} e^{i\ell\phi_k}|\ell;N\rangle. \tag{2. 38}$$

First, the states (2. 38) with different k are mutually orthogonal (by using Poisson sum rule it is easy to calculate, for proof, see Ref. [44]). They are nonseparable, and they manifest complete entanglement. Below we will use definition of complete entanglement (2. 25) and its equivalent form (2. 28) to show that the states (2. 38) manifest complete entanglement. For simplicity, we restrict examples by qubits and qutrits. Generalization for the cases of $n \geq 4$ can be constructed in a similar way.

2.2.2 Generic Entanglement

Let us first note that the generic entangled states of two and three qubits, namely the Bell and GHZ states are expressed in terms of the homogeneous states:

$$\begin{aligned} |\psi_{Bell}\rangle &= \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle), \\ |\psi_{GHZ}\rangle &= \frac{1}{\sqrt{2}}(|0,0,0\rangle \pm |1,1,1\rangle). \end{aligned}$$

To illustrate this fact, consider first the case of N qubits ($n = 2$). Then, there are only two eigenvalues of the $su(2)$ phase, namely $\phi_0 = 0$ and $\phi_1 = \pi$, so that the states (2. 38) take the form

$$|\psi^{(\pm)}_{N,2}\rangle = \frac{1}{\sqrt{2}}(|0;N\rangle \pm |1;N\rangle). \tag{2. 39}$$

At $N = 2$ and $N = 3$, it coincides with the Bell and GHZ states, respectively.

The local observables for qubits are provided by the Pauli operators

$$\sigma_x = |0\rangle\langle 1| + H.c., \quad \sigma_2 = -i|0\rangle\langle 1| + H.c., \quad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{2. 40}$$

It can be easily seen that the states (2. 39) obey the condition of complete entanglement with the observables (2. 40) for all $N \geq 2$. $\sigma_{x,y}$ just spoils the homogeneity of the states and since number of zeros and ones are equal for the phase states in the case of qubits, σ_z just equates the positive and negative ones. Hence, these states can be considered as the generic entangled states of N qubits.

It should be stressed that there are only two independent phase states (2. 39) in the case of qubits, while the dimension of the space $H_{N,2}$ is 2^N. However, beginning with the states (2. 39), one can construct a basis of completely entangled states in $H_{N,2}$ in the following way. Consider a local cyclic permutation operator ϵ_n, which in the case of qubits ($n = 2$) coincides with σ_x in (2. 40). Then, acting by this operator ϵ_2 on the individual components of the generic states (2. 39) $(2^N - 2)$ times, we get the whole basis.

For example, at $N = 2$, acting by $\epsilon_2 = \sigma_x$ on the first part in the Bell states, we get EPR states (Bell states with different phases)

$$\epsilon^{(1)}_2|\psi^{(\pm)}_{2,2}\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle \pm |0,1\rangle).$$

In the case of $N = 3$, action by the local operator $\epsilon_2 = \sigma_x$ on the first, second and third parts gives the states

$$\begin{aligned}
\epsilon^{(1)}_2|\psi^{(\pm)}_{3,2}\rangle &= \frac{1}{\sqrt{2}}(|1,0,0\rangle \pm |0,1,1\rangle), \\
\epsilon^{(2)}_2|\psi^{(\pm)}_{3,2}\rangle &= \frac{1}{\sqrt{2}}(|0,1,0\rangle \pm |1,0,1\rangle), \\
\epsilon^{(3)}_2|\psi^{(\pm)}_{3,2}\rangle &= \frac{1}{\sqrt{2}}(|0,0,1\rangle \pm |1,1,0\rangle),
\end{aligned}$$

which complete (2. 39) with respect to the whole basis of completely entangled states in the eight-dimensional space $H_{3,2}$. It should be stressed that the local operation ϵ destroys neither complete entanglement nor orthogonality of the states. The latter statement follows from the fact that $\epsilon_2^+\sigma_i\epsilon_2 = \sigma_j$.

In the case of qutrits with $n = 3$, the generic ($su(2)$ phase) states (2. 38) take the form

$$|\psi^{(k)}_{(N,3)}\rangle = \frac{1}{\sqrt{3}}(|0; N\rangle + e^{i2k\pi/3}|1; N\rangle + e^{i4k\pi/3}|2; N\rangle). \qquad (2.\ 41)$$

At $N = 2$, they coincide with the completely entangled states of two qutrits have been considered in the context of quantum information processing with ternary logic in [32]. To check with the aid of condition (2. 25) that states (2. 41) manifest complete entanglement, we should choose local observables for a qutrit as the Hermitian generators of the $su(3)$ algebra are Gell-Mann matrices (2. 21). It is now a straightforward matter to show that the states (2. 41) obey the condition with the observables (2. 21). To complete the basis of completely entangled states in $H_{N,3}$, we should again use the local cyclic permutation operator, which now takes the form

$$\epsilon_3 = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 0|.$$

Taking into account that the unitary transformation ϵ transforms any observable from (2. 21) into another observable from the same set

$$\epsilon_3^+\mathcal{O}_i\epsilon_3 = \mathcal{O}_j,$$

we can conclude that the use of ϵ_3 does not influence the complete entanglement of the generic states.

Generic states of qunits with $n \geq 4$ can be constructed in the same way.

As a result, we have shown that the generic entangled states of qunits have the form of the $su(2)$ phase states of dimension n in the basis of homogeneous states (2. 35). The basis of completely entangled states in $H_{N,n}$ can be constructed from the generic states through the use of local cyclic permutation operator.

Besides that, the consideration of the $su(2)$ algebra in the basis of homogeneous N-qunit states and its polar decomposition opens the way to define the generic entangled states as the eigenstates of certain Hermitian operators. In particular, they are eigenstates of the cosine and sine of the $su(2)$ phase operators

$$C = (E + E^{+})/2, \quad S = (E - E^{+})/2i \tag{2. 42}$$

as well as of the Hermitian phase operator

$$\Phi = \sum_k \phi_k |\psi^{(k)}_{(N,n)}\rangle\langle\psi^{(k)}_{(N,n)}|. \tag{2. 43}$$

These operators can be interpreted as the physical Hamiltonians, whose eigenstates manifest complete entanglement. Also some other Hamiltonians can be constructed through the use of local cyclic permutation operators.

For example, in the case of two qubits ($N = 2$ and $n = 2$), the operators (2. 42) and (2. 43) take the form

$$C = \frac{1}{2}(\sigma_x^{(1)} \otimes \sigma_x^{(2)} - \sigma_y^{(1)} \otimes \sigma_y^{(2)}), \quad S = 0, \quad \Phi = \pi(\mathbf{1} - C).$$

In the more interesting case of two qutrits we get

$$C = O_4^{(1)} \otimes O_4^{(2)} + \mathcal{O}_5^{(1)} \otimes \mathcal{O}_5^{(2)} + \mathcal{O}_6^{(1)} \otimes \mathcal{O}_6^{(2)} - O_7^{(1)} \otimes O_7^{(2)} - \mathcal{O}_8^{(1)} \otimes \mathcal{O}_8^{(2)} - \mathcal{O}_9^{(1)} \otimes \mathcal{O}_9^{(2)},$$

and so on.

2.3 Summary

In this chapter, we have first discussed how we can detect and measure entanglement. We have explained some entanglement measures, and introduced variance as a measure of entanglement. This measure works well for pure states of any $n-$dimensional Hilbert space.

Then, we have given an algebraic way to construct generic entangled states of qunits. We can find the whole set of completely entangled states in the corresponding Hilbert space. Maximal entanglement is necessary for some quantum information protocols, also we can achieve other entangled states by them with the help of SLOCC. We have also introduced some hamiltonians, whose eigenstates are generic entangled states.

Chapter 3

Single Particle Entanglement

In this chapter, using single spin-1 object as an example, a recent approach to quantum entanglement [30] is discussed. Within the model example under consideration, existence of single-particle entanglement is argued. The principle difference between the spin coherent and spin squeezed states, and their relation with entanglement are shown. A number of physical examples are considered.

3.1 Entanglement and nonlocality

The substance of entanglement still remains unclear, especially beyond the simplest case of two-qubit systems. In this chapter, our aim is to discuss the physics behind the quantum entanglement.

As mentioned in the previous chapter, entanglement is usually associated with quantum *nonlocality* or violation of classical realism [10, 11, 45]. Physically this is caused by the quantum correlations between the parts of the system [11]. Once created, those correlations keep on existing even after the spatial separation of parts.

On one hand, the nonlocality is probably the main distinguishing feature of quantum mechanics from classical physics. On the other hand, this notion does not contain any quantification of distance between separated entangled parts of a quantum system. Thus, it seems to be natural to assume that quantum system with strongly correlated intrinsic parts may manifest entanglement independent of distance between the parts and hence even as a local object without spatial separation of parts [6, 28, 30, 34, 46].

The quantum nonlocality is often expressed in terms of violation of different Bell-type conditions of classical realism [11]. This violation is a characteristic feature of entanglement in two-qubit systems. However, unentangled states of some systems beyond two qubits can also manifest the violation of those conditions [8, 47, 48]. For example, the difference between entangled and unentangled states disappears for systems with dynamic symmetry group $\mathrm{SU}(\mathcal{H})$ with dimension of the Hilbert space $\dim \mathcal{H} \geq 3$ (see Ref. [49], cf. [13]). As a matter of fact, violation of Bell-type conditions generally indicates the absence of “hidden” classical variables in quantum mechanics [11] rather than entanglement (also see next chapter).

This allows us to conclude that nonlocality and violation of classical realism alone are not the essential sign of entanglement and that there is no physical prohibition for the existence of entanglement of local objects (particles) caused by quantum correlations of their *intrinsic degrees of freedom*[6, 28, 30, 34].

We discussed the nonseperability criterion for the detection of entanglement, and saw that it does not work well beyond two partite systems.

It seems reasonable to focus attention on physical manifestations of entanglement in the process of measurement of quantum observables. While discussing single particle entanglement, we measure entanglement by means of variance (2. 29). Here we should again note that, physical observables defining the system plays a crucial role on entanglement.

There are several ways to understand a single particle entanglement. For example, single photon entanglement has been discussed by several groups [46, 50]. In this case a single photon goes through a 50/50 beam splitter and it is either reflected or transmitted. Initial and final states can be written as follows

$$|\Psi_{in}\rangle = |1\rangle,$$
$$|\Psi_{out}\rangle = \frac{1}{\sqrt{2}}(|1_{out1}0_{out2}\rangle + |0_{out1}1_{out2}\rangle.$$

Although there is only one photon, entanglement is between the impulse (photon) and the polarization of the photon, i.e. between the intrinsic and extrinsic degrees of freedom of the single particle.

Our concept of the single-particle entanglement considers particle itself independent of its environment. In this case, quantum correlations peculiar to entanglement can be associated with *intrinsic degrees of freedom* of the particle [6, 28, 30, 34, 46].

3.2 SU(2) qutrit

Definition of complete entanglement (2. 25) and its equivalent form (2. 28) do not assume the multipartite character of quantum systems. Does the single qubit obey the condition (2. 28)? The answer is no. The point is that the pure single-qubit state

$$\psi = a|\uparrow\rangle + b|\downarrow\rangle, \quad |a|^2 + |b|^2 = 1$$

is in fact characterized by only two real parameters ($|a|$ and $\arg a - \arg b$), for which three Eqs. (2. 28) with Pauli matrices as basic observables have only trivial solution.

For decades, qubits remain the main object of quantum information. Therefore, nonexistence of single-qubit entanglement is frequently used as a general argument against the single-particle entanglement (see Ref. [46]).

We now turn to the qutrit (2. 20), which is specified by five real parameters. Equations (2. 28) with eight basic observables (2. 21) clearly have only trivial solutions, so that, like single qubit system, single true qutrit system does not manifest entanglement.

$$|\psi\rangle = \sum_{s=-1}^{1} \psi_s |s\rangle, \qquad \sum_{s=-1}^{1} |\psi_s|^2 = 1.$$

Situation changes qualitatively if qutrit (2. 20) is considered as a state of spin-qutrit system with only three basic observables (2. 22) [6]. In this case, equations (2. 28) with three spin-1 operators (2. 22) have nontrivial solutions, so that complete entanglement of a single spin qutrit system is allowed.

In particular, it is straightforward to calculate the measure (2. 29) for the single spin-qutrit state. Taking into account that the amount of entanglement is given for an arbitrary pure single $SU(2)$ qutrit state by the expression (see Appendix) [6]

$$\mu(\psi) = 2|\psi_{-1}\psi_1 - \psi_0^2/2|. \tag{3. 1}$$

Thus, the state (2. 20) of a single spin-1 system manifests entanglement if its coefficients obey the condition

$$\frac{1}{4} \geq |\psi_{-1}|^2|\psi_1|^2 + \frac{1}{4}|\psi_0|^4 - |\psi_{-1}||\psi_1||\psi_0|^2 \cos(\phi_{-1} + \phi_1 - 2\phi_0) > 0. \tag{3. 2}$$

Here $\phi_\ell = \arg \psi_\ell$. Complete entanglement is achieved when this form (3. 2) takes the value 1/4. For example, the states

$$|\psi_0\rangle = |0\rangle \tag{3. 3}$$

and

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |-1\rangle) \tag{3. 4}$$

are completely entangled qutrit states of a single spin-qutrit system.

Before we begin to discuss the precise meaning of the above obtained result, let us mention the *relativity of entanglement* with respect to dynamic symmetry of physical system. The same state (2. 20) is unentangled if dynamic symmetry of the system is G = SU(3) and entangled in the case of reduced dynamic symmetry G′ = SU(2).

To interpret entanglement of single spin-qutrit system, let us compare it with two-qubit entanglement that has been scrutinized thoroughly.

At the beginning, we have stated that the single-particle entanglement is caused by quantum correlations between intrinsic degrees of freedom of the particle. The general picture of those correlations can be revealed through the use of well known formal correspondence between the states of single spin-qutrit and two qubits, in other words, of two spin-1/2 and single spin-1. This correspondence is given by the Clebsch-Gordon decomposition (by definition of Majorana [51] every spin-s system can be written as $2s$ spin-1/2 system):

$$\mathcal{H}_2 \otimes \mathcal{H}_2 = \mathcal{H}_3 \oplus \mathcal{H}_0, \tag{3. 5}$$

Here $\mathcal{H}_2$ denotes the two-dimensional Hilbert space of states of a single spin-$\frac{1}{2}$, $\mathcal{H}_3$ is the three-dimensional Hilbert space of spin-1, corresponding to the symmetric triplet of states in the basis of $\mathcal{H}_2 \otimes \mathcal{H}_2$, while $\mathcal{H}_0$ corresponds to the antisymmetric singlet in the basis of $\mathcal{H}_2 \otimes \mathcal{H}_2$. Denoting the basis in $\mathcal{H}_2$ by $|\uparrow\rangle$ and $|\downarrow\rangle$, we obtain the basis in $\mathcal{H}_3$ in the following form

$$|s\rangle = \begin{cases} |\uparrow\uparrow\rangle, & \text{projection of total spin } s = 1 \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + \downarrow\uparrow\rangle), & \text{projection of total spin } s = 0 \\ |\downarrow\downarrow\rangle, & \text{projection of total spin } s = -1 \end{cases} \tag{3. 6}$$

while the antisymmetric singlet is

$$|A\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{3. 7}$$

If we now assume that the singlet state (3. 7) is forbidden because of some physical reasons, then the system of two qubits becomes exactly equivalent to the spin-qutrit system. Let us stress that in some two-qubit systems the antisymmetric state is not allowed. An example of some considerable interest is provided by the so-called biphoton (photon twins created at once and propagating in the same direction) [52], where the presence of the antisymmetric state is forbidden by the requirement of symmetry of Bosonic states with respect to permutation of particles. In this case, the two qubits correspond to the polarization of photons. Another example is given by the system of two two-level atoms interacting by dipole forces in the Dicke-Lamb limit [53]. Note, that one of the symmetric states in (3. 6) is completely entangled in the two-qubit sector. This state is clearly equivalent to the state (3. 3), which is completely entangled in the spin-qutrit sector as well. On making the further assumption that spin-qutrit is a local object (particle), we have to associate the two qubits with intrinsic degrees of freedom of this object.

Thus, the single spin-qutrit entanglement can be interpreted in terms of quantum correlations between the two *intrinsic* qubits under the following conditions:
1. The Hilbert space of two qubits does not contain antisymmetric states.
2. System of two qubits is a local one, so that we can neglect the spatial separation of the qubits and thus interpret them as intrinsic degrees of freedom of a single "particle".

In the case of a single $SU(2)$ qutrit under consideration, the basic observables are given by the three spin-1 operators. As written before, in the basis $|s\rangle$, $s = \pm 1, 0$ they have the form (2. 22).

To stress the formal connection between the single $SU(2)$ qutrit and two qubits defined in the symmetric triplet subspace of $H_2 \otimes H_2$, we now note the similarity between the basic observables for two qubits and spin-1 operators (2. 22). For a single qubit, the basic observables are given by the Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3. 8}$$

defined in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, spanning the space H_2. Their representation in the whole four-dimensional Hilbert space $H_2 \otimes H_2$ for the A and B parties of the system have the form

$$\sigma_j^{(A)} = \sigma_j \otimes \mathbf{1}, \quad \sigma_j^{(B)} = \mathbf{1} \otimes \sigma_j, \quad j = x, y, z,$$

Going over from the basis

$$|\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle,$$

to the basis $\{|s\rangle, |A\rangle\}$, for the Pauli operators with $j = x$ in parties A and B we get

$$\sigma_x^{(A)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \qquad \sigma_x^{(B)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

It is seen that the only difference between $\sigma_x^{(A)}$ and $\sigma_x^{(B)}$ consists in the form of the column and row, corresponding to the antisymmetric state $|A\rangle$, while the (3×3) principle submatrices coincide with each other and with the S_x operator in Eq. (2. 22). Thus, discarding the antisymmetric singlet state, we reduce both local observables $\sigma_x^{(A)}$ and $\sigma_x^{(B)}$ to the same spin-1 operator S_x. The same result can be obtained for other observables as well.

In view of the condition (2. 28), one can conclude that the spin-1 states of a single $SU(2)$ qutrit

$$\frac{1}{\sqrt{2}}(|+1\rangle \pm |-1\rangle), \quad |0\rangle \tag{3. 9}$$

are completely entangled states. It is clear that they are also completely entangled in the symmetric triplet sector of the two-qubit Hilbert space. The fact that the spin-1 state $|0\rangle$ with $s = 0$ is completely entangled seems to be quite interesting. Physical Interpretation of such an entanglement will be discussed later.

Any of the states (3. 9) can be used as the generic entangled state. Consider for example the state $|0\rangle$ and SLOCC of the form

$$\exp(zS_x) = \frac{1}{2} \begin{pmatrix} \frac{e^z+e^{-z}}{2} + 1 & \frac{e^z-e^{-z}}{\sqrt{2}} & \frac{e^z+e^{-z}}{2} - 1 \\ \frac{e^z-e^{-z}}{\sqrt{2}} & e^z + e^{-z} & \frac{e^z-e^{-z}}{\sqrt{2}} \\ \frac{e^z+e^{-z}}{2} - 1 & \frac{e^z-e^{-z}}{\sqrt{2}} & \frac{e^z+e^{-z}}{2} + 1 \end{pmatrix},$$

where z is an arbitrary complex number. We get

$$\exp(zS_x)|0\rangle = \frac{e^z + e^{-z}}{2}|0\rangle + \frac{e^z - e^{-z}}{2\sqrt{2}}(|+1\rangle + |-1\rangle). \tag{3. 10}$$

One can easily see that the state, obtained from Eq. (3. 10) by proper normalization, always manifests nonzero entanglement. At any imaginary z, there is complete entanglement in the system. In the opposite case of real Z we get

$$\mu(\exp(zS_x)|0\rangle) = \frac{1}{\cosh(2Re(z))},$$

so that $\mu(\exp(zS_x)|0\rangle) \in [1, 0)$ at $Re(z) \in [0, \infty)$. The complete entanglement is achieved here only at $Re(z) = 0$, when SLOCC coincides with the identity operator.

3.3 Entanglement and quantum fluctuations

Entanglement generally manifests itself by means of specific behavior of quantum uncertainties. Thus, an idea to compare it with other phenomena defined in terms of quantum uncertainties clearly suggests itself. It seems to be natural to compare entangled, coherent, and squeezed states of the same system.

For example, Glauber coherent state of Bose fields [54] manifests the minimal amount of quantum fluctuations of the field quadratures. Its generalization on the case of spin-like systems [55] is also characterized by the minimal amount of quantum uncertainties (also see Refs. [56]). The generalized coherent states [57, 58] can be defined as the states of minimal uncertainty.

In turn, the squeezed states of Bose field [59] assume that the uncertainty of one of the field quadratures is lower than the minimal uncertainty, while another quadrature has quite high uncertainty. The same idea is used in the definition of spin squeezed states [60]. Namely, according to Ref. [60] squeezing corresponds to the decrease of uncertainty of either spin component S_x and S_y below the value $\frac{1}{2}|\langle[S_x, S_y]\rangle|$, whose square gives the right-hand side of the Heisenberg uncertainty relation. It has been noticed in Ref. [61] that similar behavior can be observed for the spin coherent states as well. Therefore, it has been proposed to associate spin squeezing with certain correlations between parties in multi-spin systems [61]. In fact, these correlations can be similar to the ones responsible for the formation of entangled states [62] (for further discussion of spin squeezed states, see Refs. [63] and references therein).

In view of manifestation of conventional squeezing of quantum uncertainties below *standard quantum limit* by both coherent and "squeezed" spin states [61], hereafter we use the term "squeezed" in the context of spin states in quotation marks.

It is known that an entangled two-qubit state is associated with the SU(2) squeezed states [63], while unentangled states are the SU(2) coherent states [8, 47]. We will show that this interpretation is valid for the entangled and unentangled states of a single spin-qutrit as well.

3.3.1 The SU(2) coherent states are unentangled

The Glauber coherent state of Bose field [54] is defined by action of the unitary displacement operator

$$D(\alpha) = \exp(\alpha a^{+} - \alpha^{*} a) \tag{3. 11}$$

on the vacuum state $|vac\rangle$:

$$|\alpha\rangle = D(\xi)|vac\rangle.$$

Here a and a^{+} denote the annihilation and creation operator for the Bose field under consideration and vacuum state obey the stability condition $a|vac\rangle = 0$.

It is generally accepted that the $SU(2)$ version of Glauber coherent states is defined in the similar fashion [55, 56] (for review, see Ref. [64]. Namely, first we have to introduce the rising and lowering spin operators

$$S_{+} = S_x + iS_y, \quad S_{-} = S_x - iS_y \tag{3. 12}$$

for an arbitrary spin. Then, the $SU(2)$ spin coherent state $|\alpha\rangle$ is defined by action of the displacement operator

$$D(\alpha) = \exp(\alpha S_{+} - \alpha^{*} S_{-}), \quad \alpha \in C, \tag{3. 13}$$

on the state $|-s\rangle$:

$$|\alpha\rangle = D(\alpha)|-s\rangle. \tag{3. 14}$$

This state $|-s\rangle$ is considered as an analogue of the vacuum state $|vac\rangle$ because $S_-|-s\rangle = 0$:

In the "vacuum" state $|-s\rangle$, the spin has a given projection $-s$ onto the z-axis $\langle -s|S_z|-s\rangle = -s$, so that the corresponding variance $V(S_z; -s) = 0$. For the two other spin operators in the direction orthogonal to the quantization axis z we get

$$\langle -s|S_x|-s\rangle = \langle -s|S_y|-s\rangle = 0,$$
$$V(S_x; -s) = \langle -s|(\frac{S_+ + S_-}{2})^2|-s\rangle = s/2$$
$$V(S_y; -s) = \langle -s|(\frac{S_+ - S_-}{2i})^2|-s\rangle = s/2,$$

so that the total variance (2. 24) takes the from

$$V(-s) = s.$$

This is the minimal value of the total variance for the spin-s system under consideration. Thus, in view of the definition of entanglement, given in the previous Section, the state $|-s\rangle$ is unentangled.

According to Eq. (2. 27), the maximum of the total variance of a single spin-s system is

$$V_{max} = V(\psi_{CE}) = s(s+1).$$

This allows us to represent the measure of entanglement (2. 29) for a single spin-s system in the following form

$$\mu(\psi) = \frac{1}{s}\sqrt{V(\psi) - s}. \tag{3. 15}$$

Thus, the measure (2. 29) vanishes for coherent states.

It is easily seen that, in the case of a single qubit ($s = 1/2$), any state of the system is a coherent one.

For the spin-1 operators (3), the rising and lowering operators (10) take the form

$$S_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Therefore, the displacement operator (3. 13) is represented as follows

$$D(\alpha) = \begin{pmatrix} \frac{1+\cos 2|\alpha|}{2} & \frac{e^{i\phi}\sin 2|\alpha|}{\sqrt{2}} & \frac{e^{2i\phi}(1-\cos 2|\alpha|)}{2} \\ -\frac{e^{-i\phi}\sin 2|\alpha|}{\sqrt{2}} & \cos 2|\alpha| & \frac{e^{i\phi}\sin 2|\alpha|}{\sqrt{2}} \\ \frac{e^{-2i\phi}(1-\cos 2|\alpha|)}{2} & -\frac{e^{-i\phi}\sin 2|\alpha|}{\sqrt{2}} & \frac{1+\cos 2|\alpha|}{2} \end{pmatrix}.$$

Here $\phi = \arg\alpha$. It is now a straightforward matter to arrive at the relation

$$\begin{aligned} |\alpha\rangle \equiv D(\alpha)|-1\rangle = \frac{e^{2i\phi}}{2}[1-\cos(2|\alpha|)]|+1\rangle \\ +\frac{e^{i\phi}}{\sqrt{2}}\sin(2|\alpha|)|0\rangle + \frac{1}{2}[1+\cos(2|\alpha|)]|-1\rangle. \end{aligned} \tag{3. 16}$$

It is evident that the measure (3. 1) has zero value for the state (3. 16) at any α like in the case of state $|-1\rangle$. This is natural. The point is that the operator $(\alpha S_+ - \alpha^* S_-)$ in (3. 11) belongs to the $su(2)$ algebra, so that the displacement operator (3. 11) amounts to an SU(2) rotation. This means that every spin coherent state (3. 16) is just a state with minimal spin projection $-s$ onto some direction, which can be chosen as a new quantization axis. Thus, there is no principle difference between the spin coherent state and state $|-s\rangle$. In particular, spin coherent state is as unentangled as the state $|-s\rangle$.

Let us calculate the expectation values of the basic observables (3. 5) in the state (3. 16)

$$\begin{aligned} \langle\alpha|S_x|\alpha\rangle &= \sin(2|\alpha|)\cdot\cos\phi, \\ \langle\alpha|S_y|\alpha\rangle &= -\sin(2|\alpha|)\cdot\sin\phi, \\ \langle\alpha|S_z|\alpha\rangle &= -\cos(2|\alpha|). \end{aligned} \tag{3. 17}$$

The corresponding uncertainties have the form

$$\begin{cases} V_x(\alpha) = \frac{1}{2}[1-\sin^2(2|\alpha|)\cos^2\phi] \\ V_y(\alpha) = \frac{1}{2}[1-\sin^2(2|\alpha|)\sin^2\phi] \\ V_z(\alpha) = \frac{1}{2}\sin^2(2|\alpha|) \end{cases} \tag{3. 18}$$

so that the total uncertainty for the spin coherent state (13) is

$$V(\alpha) = V_x(\alpha) + V_y(\alpha) + V_z(\alpha) = 1,$$

for any $\alpha \in C$. As expected, this is the minimal value of $V(\psi)$ in the case of a single $SU(2)$ qutrit with basic observables (2. 22). It is seen that the maximal value of the total uncertainty given by the Casimir operator $S_x^2 + S_y^2 + S_z^2 = 2 \times \mathbf{1}$ is $V_{max} = 2$.

In the sector of two qubit,

$$S_\pm = \frac{1}{2}\left[\tilde{\sigma}_\pm^{(A)} + \tilde{\sigma}_\pm^{(B)}\right],$$

where $\tilde{\sigma}$ denotes the corresponding operator acting in the symmetric part H_3 of the two-qubit Hilbert space $H_2 \otimes H_2$. Since $[\tilde{\sigma}^{(A)}, \tilde{\sigma}^{(B)}] = 0$, the displacement operator (3. 13) is factorized in the qubit representation

$$D(\alpha) = D^{(A)}(\alpha)D^{(B)}(\alpha),$$

where

$$D^{(A,B)}(\alpha) = \exp(\alpha\tilde{\sigma}_+^{(A,B)}/2 - \alpha^*\tilde{\sigma}_-^{(A,B)}/2).$$

Thus, the coherent state (3. 14) can be considered as the separable state

$$D(\alpha)|-1\rangle = D^{(A)}(\alpha)D^{(B)}(\alpha)|\downarrow\downarrow\rangle$$

of the intrinsic degrees of freedom of the $SU(2)$-qutrit particle.

In the case of Bose field, coherent states realize exact equality in the Heisenberg uncertainty relation [57, 58, 64]. For the spin systems, the uncertainty relation usually considered in the context of coherence has the form [55, 56]

$$V_x(\psi)V_y(\psi) \geq \frac{1}{4}|\langle\psi|S_z|\psi\rangle|^2. \quad (3.\ 19)$$

It follows from Eq. (3. 18) that exact equality in (3. 19) is achieved under the condition $\phi = k\pi$, $k = 0, 1, \cdots$. It is also seen that under this condition

$$V_x(\alpha) \leq \frac{1}{2}|\langle\alpha|S_z|\alpha\rangle|.$$

We now note that according the definition given in Ref. [60] the inequality

$$V_j(\psi) < \frac{1}{2}|\langle\psi|S_z|\psi\rangle|, \quad j = x, y. \tag{3. 20}$$

corresponds to the spin squeezed states. Thus, unlike the case of Bose field, the spin coherent state (13) manifests squeezing of quantum uncertainties below the so-called *standard quantum limit*. More detailed discussion of this fact we postpone to next sections.

3.3.2 “Squeezed” spin states

To avoid difficulties caused by the fact the the $SU(2)$ coherent state can manifest squeezing of quantum uncertainties, it has been proposed to construct spin “squeezed” states in the same fashion as the conventional Bose squeezed states. The latter are defined by action of the unitary squeeze operator [59]

$$\mathcal{S}(\xi) = \exp\left[\frac{1}{2}\left(\xi^* a^2 - \xi a^{+2}\right)\right], \quad \xi \in C \tag{3. 21}$$

on the vacuum state. Following Ref. [61], the spin “squeezed” state can be defined in direct analogy with the case of Bose field as follows

$$|\xi\rangle = \mathcal{S}(\xi)|-s\rangle = \exp\left[\frac{1}{2}\left(\xi^* S_-^2 - \xi S_+^2\right)\right]|-s\rangle. \tag{3. 22}$$

In the case of single $SU(2)$ qutrit under consideration $|-s\rangle = |-1\rangle$.

It is clear that such a state cannot be defined for a single spin-$\frac{1}{2}$ system (qubit) because the Pauli rising and lowering operators obey the condition $\sigma_\pm^2 = 0$.

In the case of spin-1 system under consideration, the squares of the rising and lowering operators are

$$S_+^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_-^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

so that the squeeze operator (3. 22) takes the form

$$\mathcal{S}(\xi) = \begin{pmatrix} \cos|\xi| & 0 & -e^{i\varphi}\sin|\xi| \\ 0 & 1 & 0 \\ e^{-i\varphi}\sin|\xi| & 0 & \cos|\xi| \end{pmatrix}, \tag{3. 23}$$

where $\varphi = \arg \xi$. Then, the "squeezed" state (3. 22) of a single $SU(2)$ qutrit is represented as follows

$$\begin{aligned} |\xi\rangle &= \mathcal{S}(\xi)|-1\rangle \\ &= -e^{i\varphi} \sin|\xi||+1\rangle + \cos|\xi||-1\rangle. \end{aligned} \tag{3. 24}$$

It is seen that the measure (3. 1) is always greater than zero for the state (3. 24)

$$\mu(\xi) = |\sin(2|\xi|)|$$

except the points $|\xi| = k\pi/2$, $k = 0, 1, \cdots$, in which the state (3. 24) coincides with either $|+1\rangle$ or $|-1\rangle$ unentangled state. Thus, the spin-1 "squeezed" state manifests entanglement at $|\xi| \neq k\pi/2$. At $\xi = \pi/4 + k\pi$, the state (3. 24) coincides with the first two states in equation (3. 9).

This is not an unexpected result. In fact, the exponent in Eq. (3. 22), being rewritten in terms of Pauli operators for intrinsic qubit degrees of freedom,

$$\mathcal{S}^{(AB)} = \exp\left[\frac{1}{2}(\xi^* \tilde{\sigma}_-^{(A)} \tilde{\sigma}_-^{(B)} - \xi \tilde{\sigma}_+^{(A)} \tilde{\sigma}_+^{(B)})\right], \tag{3. 25}$$

which reminds the squeeze operator for two-modes Bose field [65]. Note that the expression

$$\xi^* \tilde{\sigma}_-^{(A)} \tilde{\sigma}_-^{(B)} - \xi \tilde{\sigma}_+^{(A)} \tilde{\sigma}_+^{(B)}$$

can be interpreted as Hamiltonian whose eigenstates are completely entangled [41]. For relation between the spin "squeezing" and entanglement, also see Refs. [62, 63].

Performing averaging over "squeezed" state (3. 24), for the observables (2. 22) we get

$$\langle\xi|S_{x,y}|\xi\rangle = 0, \quad \langle\xi|S_z|\xi\rangle = -\cos(2|\xi|). \tag{3. 26}$$

In turn, the corresponding uncertainties are

$$\left\{ \begin{array}{lcl} V_x(\xi) & = & [1 - \sin(2|\xi|)\cos\varphi]/2 \\ V_y(\xi) & = & [1 + \sin(2|\xi|)\cos\varphi]/2 \\ V_z(\xi) & = & \sin^2(2|\xi|) \end{array} \right\} \tag{3. 27}$$

Thus, the total uncertainty in the spin "squeezed" state (3. 24) takes the value

$$V(\xi) = 1 + \sin^2(2|\xi|),$$

which exceeds the minimal value $V_{min} = 1$ at $\xi \neq k\pi/2$.

It is seen that under the condition $\varphi = k\pi$ the uncertainty relation (3. 19) becomes exact equality for the state (3. 24) and that one of the uncertainties $V_{x,y}(\xi)$ obey the condition of squeezing (3. 20) except the points $|\xi| = \frac{\pi}{4} + k\pi$, corresponding to the completely entangled states. The latter cannot be considered as squeezed in the sense of definition (3. 20) because $\langle S_z \rangle = 0$ in this case, while $V_{x,y} \geq 0$. It is also seen that the states (3. 24) do not obey the condition of squeezing (3. 20) at $\varphi = \frac{\pi}{2} + k\pi$ and an arbitrary $|\xi|$. Thus, the so-called spin "squeezed" states can violate the condition of squeezing (3. 20).

3.3.3 Spin coherence and "squeezing"

We have shown in the two previous Sections that it is impossible to distinguish between the spin coherent (3. 16) and spin "squeezed" (3. 24) states through the use of condition (3. 20), specifying quantum fluctuations beyond the so-called standard quantum limit. That is why we write the term *spin "squeezed" state* using the quotation marks.

The qualitative difference between these two states is characterized by their relation to the entanglement. Namely, spin coherent states (3. 16) are always unentangled, while the spin "squeezed" states are entangled (except a null set of points $\xi = k\pi$, $k = 0, 1, \cdots$. In other words, they are non-equivalent with respect to SLOCC.

This distinction is agreed with the main idea of Ref. [61] that spin squeezed state should contain certain quantum correlations. In fact, in Ref. [61], a spin $S \geq 1$ is considered as a collective system of "elementary" spins $\frac{1}{2}$, and squeeze operator similar to that used in Eq. (3. 22) establishes correlations between the elementary parties of the system. These correlations are responsible for entanglement of the corresponding state [62, 63].

The use of the total uncertainty makes it possible to quantify the difference between the states (3. 16) and (3. 24). Namely, for spin coherent state (3. 16) $V(\alpha) = V_{min}$ for all $\alpha \in C$, while for spin "squeezed" state (3. 24) $V(\xi) > V_{min}$.

This difference in behavior of total uncertainty has direct connection with the measure of entanglement (3. 1). In the case of the $SU(2)$ qutrit under consideration, $V_{max} - V_{min} = 1$, so that the measure has the form

$$\mu(\psi) = \sqrt{V(\psi) - V_{min}}.$$

Thus, the deviation of the total uncertainty with respect to its minimal value, specifying the difference between spin coherent (3. 16) and spin "squeezed" (3. 24) states, coincides with the natural measure of entanglement.

In view of the above discussion, it seems to be natural to rename spin "squeezed" states (3. 22) and to call them either *correlated* states (in the spirit of philosophy of Ref. [61]) or just entangled states.

Following Kitagawa and Ueda [61], we call spin state ξ to be squeezed iff $V_r(\xi) < s/2$ for some direction $r \perp \vec{s}$, where

$$\vec{s} = \vec{e}_x \langle S_x \rangle + \vec{e}_y \langle S_y \rangle + \vec{e}_z \langle S_z \rangle$$

is the direction of the average spin vector.

This means that in a coordinate system with the z-axis along the average spin vector $\vec{s}$, we have $\langle S_x \rangle = \langle S_y \rangle = 0$, $\langle S_z \rangle = \pm s$, and $V_z(\xi) = 0$. So we can write

$$V_x(\xi) + V_y(\xi) \geq s \qquad (3.\ 28)$$

in contrast to the spin-coherent state. It is easy to check that this condition of squeezing (3. 28) is valid for the states (3. 3) and (3. 4), therefore they are squeezed.

Conventional picture of squeezing [59, 60] assumes a certain skewness of quantum uncertainties and their transformation from circles, corresponding to coherent states, to ellipses [59, 65]. According to Kitagawa and Ueda [61], for the coherent states, uncertainties in the plane orthogonal to a given state correspond to a circle in the orthogonal plane, while for the squeezed states those uncertainties corresponds to an ellipse of uncertainties.

Consider the equation (3. 17), if we transform our reference frame to make $\vec{s}$ along z-axis. Turning to the rotated frame, we can check that the variances of basic observables in the plane orthogonal to $\vec{s}$ form a circle. According to reference [61], this means that the state $|\alpha\rangle$ does not manifest squeezing.

Now, if we look at the equation (3. 26), direction of the $\vec{s}$ is along the z-axis. one of the variances of S_x and S_y is always smaller than $1/2$, corresponding to criterion of squeezing of Ref. [61], except the points $\varphi = \pi/2 + k\pi$ when $V_x(\xi) = V_y(\xi) = 1/2$. Note that changing the phase φ amounts to a rotation in xy plane. By choosing $\varphi = 0$ we get the extremal values of V_x and V_y

$$V_x(\xi) = \frac{1 - \sin(2|\xi|)}{2}, \quad V_y(\xi) = \frac{1 + \sin(2|\xi|)}{2},$$

which are just squares of semi-axes of the uncertainty ellipse.

This picture corresponds to the representation of states in a spherical phase space. In the case of spin-$\frac{1}{2}$ states, this is the Bloch sphere (e.g., see Ref. [64]). If we return to the interpretation of single $SU(2)$ qutrit as two qubits, that has been discussed in previous sections, then it can be easily seen that (3. 16) represents a separable state

$$\begin{aligned} |\alpha\rangle = [&e^{i\phi}\sin(|\alpha|)|\uparrow\rangle^{(A)} + \cos(|\alpha|)|\downarrow\rangle^{(A)}] \\ &\otimes[e^{i\phi}\sin(|\alpha|)|\uparrow\rangle^{(B)} + \cos(|\alpha|)|\downarrow\rangle^{(B)}], \end{aligned}$$

in which both spins $\frac{1}{2}$ (qubits) point in the same mean direction in the spherical phase space. Due to separability, the uncertainties for different qubits are not correlated in the coherent state. In turn, the “squeezed” state (3. 24) cannot be factorized because of bilinear nature of the squeeze operator in Eq. (3. 22) and manifests correlation of uncertainties.

3.4 Physical realizations of single spin-qutrit entanglement

Qubit systems are often associated with two-level atoms, where quantum correlations between the atoms can be generated either by photon exchange or by means of dipole-dipole interaction. In the latter case, decrease of the interatomic distance down to the lamb-Dicke limit (interatomic separation becomes much shorter than the wavelength of two-level transition) leads to an effective discard of the antisymmetric state [53]. Thus, this two-qubit system behaves like a single spin-qutrit object.

The nice feature of this example is that the reduction of symmetry

$$[\mathrm{SU}(2) \times \mathrm{SU}(2)]_{(\text{in 4 dimensions})} \to \mathrm{SU}(2)_{(\text{in 3 dimensions})}$$

and localization accompany each other.

Another example is provided by the so-called *biphoton*. With respect to polarization, this object represents the SU(2) ternary system (spin-qutrit) and is as local as a single photon. Antisymmetric state with respect to permutations is forbidden here by the Bosonic nature of photons. Undoubtedly, biphoton can be split into spatially separated photons, carrying polarization qubits. But before splitting, it should be considered as a local spin-qutrit object.

The simplest three-photon interaction Hamiltonian, describing parametric down-conversion process, has the form

$$H_{int} = g(b^+ a_H a_V + a_V^+ a_H^+ b). \tag{3. 29}$$

Here operators b, b^+ correspond to the incident photon, while the operators a_H, a_H^+ and a_V, a_V^+ describe the outgoing photons of the same frequency with Horizontal and Vertical polarizations, respectively. If we assume that the initial state contains only one incident photon $|in\rangle = |1_b\rangle \otimes |0_H, 0_V\rangle$, then

$$H_{int}|in\rangle = |out\rangle = |0_b\rangle \otimes |1_H, 1_V\rangle.$$

The $|out\rangle$ state here is a state of two photons of the form of conjoint Siamese twins that are not separated geometrically as long as they propagate along the same direction. In this case their polarization state should be symmetric as for all bosons. At the bottom of line, it represents the neutrally-polarized state $|0\rangle$ of a (quasi-)particle called biphoton. A "surgery" separating Siamese photon twins is provided by a beam splitter that transforms $|out\rangle$ state into the two-qubit states

$$\frac{1}{\sqrt{2}}(|1_H\rangle_A + |1_H\rangle_A) \otimes (|1_V\rangle_B + |1_V\rangle_B)$$

where A and B denote the orthogonal directions of photons after beam-splitter.

The whole polarization triplet of biphoton states has the form [52]

$$|+1\rangle = |2_H\rangle,\ |0\rangle = |1_H, 1_V\rangle,\ |-1\rangle = |2_V\rangle. \qquad (3.\ 30)$$

It is then easily seen that the Stokes operators

$$\begin{cases} S_x &= \frac{1}{2}(a_H^+ a_V + a_V^+ a_H) \\ S_y &= \frac{-i}{2}(a_H^+ a_V - a_V^+ a_H) \\ S_z &= \frac{1}{2}(a_H^+ a_H - a_V^+ a_V) \end{cases} \qquad (3.\ 31)$$

obey the spin commutation relations and act on the states (3. 30) as spin-1 operators.

Apologists of the standpoint that entanglement is inherent to systems with spatially separated parties can say that the above two examples do not fit the notion of a single particle. Therefore, we now turn to examples that definitely correspond to a single particle entanglement.

An important example of the SU(2) ternary system is provided by the three-level atom with λ-type transition shown in Fig. 1.

Here the highest excited level can be associated with the state $|0\rangle$ of spin 1, while the two lower levels with the states $|+1\rangle$ and $|-1\rangle$, respectively.

The Hamiltonian, describing interaction between the atom and two cavity modes, has the form

$$H_{int} = g_1 R_{0+} a_1 + g_2 R_{0-} a_2 + H.c., \qquad (3.\ 32)$$

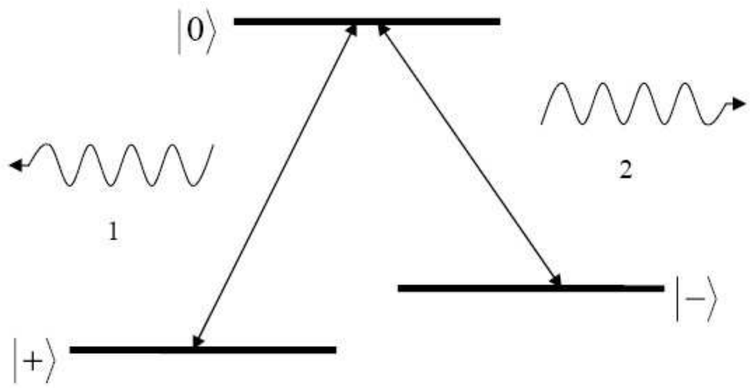

Figure 3. 1: Interaction between λ-type three-level atom and two cavity modes.

where g_i denotes the corresponding coupling constant, $R_{bc} = |b\rangle\langle c|$ is the atomic operator, and a_i is the photon annihilation operator for the field mode $i = 1, 2$. The spin operators (2. 22) have the form

$$\begin{aligned} S_x &= \frac{1}{\sqrt{2}}(R_{+0} + R_{0+} + R_{0-} + R_{-0}), \\ S_y &= \frac{1-i}{\sqrt{2}}(R_{+0} + -R_{0+} + R_{0-} - R_{-0}), \\ S_z &= R_{++} - R_{--}. \end{aligned} \tag{3. 33}$$

In view of the results of Sec. III, the state $|\psi_{in}\rangle = |0\rangle \otimes |\mathrm{vac}\rangle$ of the atom-field system, in which the atom is in excited state and cavity field is in the vacuum state, is completely entangled with respect to the atomic observables given by Eq. (3. 33). Under influence of the atom-photon interaction (4. 25), this state passes to the following normalized state

$$\frac{1}{\sqrt{g_1^2 + g_2^2}}(g_1|+\rangle \otimes |1_1\rangle + g_2|-\rangle \otimes |1_2\rangle) \tag{3. 34}$$

and vice versa. This state (3. 34) can be interpreted as the two-qubit state, where one qubit is formed by the atomic states $|\pm\rangle$ and the second qubit by the photon states $|1_1\rangle$ and $|1_2\rangle$. Clearly, this state is entangled, and the corresponding concurrence [22] has the form

$$\mu = \frac{2|g_1 g_2|}{g_1^2 + g_2^2}.$$

This example clearly illustrates decay of the single spin-qutrit entangled state $|\psi_{in}\rangle$ into the two-qubit entanglement. The atom and photon qubits can be spatially separated by cavity leakage.

Another important example of the SU(2) entanglement of single particle is provided by the *isotriplet* of π-mesons. The completely entangled isospin-1 state of π^0 meson can be interpreted as the complete entanglement of two quark qubits, corresponding to the intrinsic degrees of freedom of this single particle. Since the complete entangled states manifest the maximal amount of quantum uncertainties (in the case of pions, these are the quantum fluctuations of quarks), all one can expect is that the neutral π^0 meson should be less stable then the charged pions, which is just the case. For detailed discussion of this example we refer recent work [30].

The above example of meson isotriplet is similar to the Cooper pairs in superfluid phases of ^3He. It is well known that the atoms of ^3He have spin $s = \frac{1}{2}$ each and that the total spin of a Cooper pair is $s = 1$, so that the antisymmetric state of two atomic qubits is forbidden [66]. Note that in the BCS superconductors where $s = 0$, the only allowed pair wave function is given by the antisymmetric singlet state (3. 7).

Another simple example of a single particle with spin 1, which can manifest entanglement, is provided by the deuteron, which is a nucleus of a deuterium atom, consisting of weakly bounded proton and neutron [67]. Note that, unlike π^0 meson, this is a stable particle. Each nucleon in the deuteron can be considered as an intrinsic qubit with respect to its spin $\frac{1}{2}$. An experimental proof of the existence of entanglement in deuteron and of the possible use of it for quantum teleportation of spin states of massive particles has been reported recently Ref. [68].

It seems to be tempting to consider a photon as a single $SU(2)$ qutrit. Note that although photon spin $s = 1$, the absence of the rest mass allows only two spin states (helicities) usually associated with the photon polarization [69] (the photon polarization qubit).

At the same time, photons emitted by atomic, molecular, and nuclear transitions between the states characterized by a given value of the *total angular momentum and parity* carry these physical quantities due to the conservation laws [69, 70]. The representation of those *multipole* photons is given by quantization of spherical waves emitted by a point-like source (atom, for example) [71]. The total angular momentum of photons consists of the spin and orbital parts:

$$\vec{J} = \vec{S} + \vec{L}.$$

Photons with total angular momentum j and parity $P = (-1)^j$ are called the *electric-type j-pole* photons. Those photons have only two allowed values of the orbital angular momentum, namely $\ell = j - 1$ and $\ell = j + 1$. Thus, the orbital angular momentum of electric-type photons can also be considered as a qubit.

The case of $j = 1$ and parity $P = -1$ corresponds to the *electric dipole* ($E1$) photons, which are probably the most widespread type of photons in the universe. The quantum state of $E1$ photons contains a certain linear combination of states with $\ell = 0$ and $\ell = 2$, so that the orbital angular momentum of those photons does not have a well defined value [69]. This also means that spin (polarization) and orbital momentum are strongly correlated and that the total angular momentum cannot be divided into spin and orbital contributions.

With respect to the total angular momentum $j = 1$, a single $E1$ photon should be considered as the $SU(2)$ qutrit, whose intrinsic qubit degrees of freedom correspond to polarization and orbital angular momentum qubits.

During the last decade, the orbital angular momentum of photons has attracted a great deal of experimental interest (e.g, see Ref. [72] and references therein). In particular, entanglement of photons with respect to their orbital angular momentum has been observed [73]. The photon beams far from the source were used in these experiments. At the same time, specific features of the dipole photons and correlation between spin and orbital parts of the angular momentum should be maximally visible at short distances (less than the wavelength) where spherical waves of photons cannot be successfully approximated by plane waves.

It is possible to find many other examples, from the spin-1 atoms like ^{87}Rb and ^{23}Na, widely used in investigation of Bose-Einstein condensation, to the more exotic systems like vector mesons and three spin-1 gauge bosons in the standard model [74], in which spin-qutrit entanglement may be realized.

3.5 Summary

In this section, we have examined quantum entanglement of a single SU(2) qutrit. We have shown that the spin-1 state with projection $s = 0$ manifests complete entanglement and can be used as the generic entangled state with respect to SLOCC.

We discussed the relation between quantum fluctuations and entanglement. We have proved that the SU(2) coherent state is always unentangled and shows minimal amount of uncertainty, while spin squeezed states manifest entanglement.

We have shown a number of physical systems that realize the SU(2) qutrit states, and therefore can be prepared as the single particle entangled states.

Chapter 4

Violation of Bell type condition without nonlocality

In this chapter, a test for compatibility of local spin-1 system with hidden variables model will be presented. A variation of this test can be applied to detect entanglement in closely tight systems, where separate measurements on the components are unfeasible. As an example we consider in some details a biphoton system.

4.1 Introduction

Most striking manifestation of entanglement is *nonlocality*. To give the definition again, it can be understood as a correlation beyond light cones of spatially separated quantum systems, where no classical interaction between them is possible. However, for quantum computation a magic ability of entanglement to bypass constraints imposed by so called *classical realism* is far more important. The latter is understood here as existence of hidden parameters, or similarly a joint probability distribution of all involved quantum observables. This property of entanglement makes it impossible *in principle* to model it on any classical device, and emphasizes a *qualitative* distinction between classical and quantum information processing, not merely difference in their computational power.

After Bell's work [11], nonclassical behavior is usually detected by violation of certain inequalities [75, 76], collectively named Bell's conditions. Their experimental test [77] left little or no doubt that entangled states indeed override the classical constraints, in spite of continuing search for possible loopholes [78, 79, 80].

Initially Bell applied his analysis to a nonlocal EPR-Bohm system [10, 81]. However, the nonlocality has been used only to justify *simultaneous measurements* on remote parts of the system. The possibility of such measurements amounts to *commutativity* of the corresponding operators, that can happen in local systems as well. For example, squares of spin projections onto two orthogonal directions in a spin-1 system commute. This gives us a chance to extend Bell's approach to *local systems.*

Let's elucidate a drastic difference between coherent $S_z = 1$ and entangled $S_z = 0$ spin states. For this, following Penrose interpretation [82, p.589] of Hardy proposal [76], consider a decay of a spin-1 particle into two spin-1/2 components. The resulting two particle state should be symmetric with respect to interchange of the particles and should preserve the angular momentum. This forces the coherent state $S_z = 1$ to decay into separable state $|\uparrow\uparrow\rangle$, while $S_z = 0$ must decay into Bell state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$. The latter manifests all the surprising properties of entanglement.

The problem we address here is whether we can detect something non-classical in the state $S_z = 0$ before the decay?

Recall that a macroscopic measurement of a quantum observable X in a state $\psi \in \mathcal{H}$ results in a random quantity x (e.g., see [83]), whose numerical values coincide with eigenvalues of operator X. Its probability distribution in state ψ is implicitly determined by expectations $E(f(x)) = \langle\psi|f(X)|\psi\rangle$ for all functions $f(x)$. Joint probability distribution of *commuting* observables X, Y can be deduced from expectations $E(f(x, y)) = \langle\psi|f(X, Y)|\psi\rangle$.

The essence of the hidden variables hypothesis can be stated as follows: *all physically relevant observables in a given state ψ have a hidden joint probability distribution.* This amounts to compatibility of the partial joint distributions of *commuting* observables. Finding such compatibility conditions is known in mathematics as *marginal problem,* see [48] and references therein. It lies at the very heart of the Bell's constraints, and comes close to Wigner's analysis of hidden variables [84].

The advantage of the above approach is that it goes beyond spatially separated systems, and open a possibility for violation of the classical realism in local systems. Such violation is usually associated with entanglement [6, 47].

4.1.1 Bell's inequality

Einstein, Podolsky, and Rosen used momentum and coordinate to discuss the physical reality. Bohm and Aharonov [85] introduced spin to the discussion. Bell used singlet spin state to show the violation of classical realism. Spins are moving along opposite directions and measurements are done by Stern-Gerlach apparatus on spin components $\vec{\sigma}_1$ and $\vec{\sigma}_2$. Bell introduced parameter λ for the complete specification of the state. Result of a measurement just depends on this parameter and the direction on which measurement is done along. Suppose $A_\lambda(\hat{n}) = \pm 1$ and $B_\lambda(\hat{n}) = \pm 1$ are the results of measurements of first and second spins along the specified directions. Expectation value for product outcomes can be written as following

$$E(\hat{n}_1, \hat{n}_2) = \int A_\lambda(\hat{n}_1) B_\lambda(\hat{n}_2) \rho(\lambda) d\lambda \tag{4. 1}$$

where $\rho(\lambda)$ is the probability distribution of λ and $\int \rho(\lambda) d\lambda = 1$. Note that $A_\lambda(\hat{n}_1)$ is independent of $\hat{n}_2$ and vice versa, as required by the locality assumption. Expectation (4. 1) should be equal to quantum mechanical expectation value

$$< (\vec{\sigma}_1 \cdot \hat{n}_1)(\vec{\sigma}_2 \cdot \hat{n}_2) > . \tag{4. 2}$$

To compute (4. 2) for singlet state $|A\rangle = \frac{1}{\sqrt{2}}(|(\hat{n}+),(\hat{n}-)\rangle - |(\hat{n}-),(\hat{n}+)\rangle)$ (since there is no preferred spin direction, singlet state has rotational invariance), first notice

$$(\vec{\sigma}_1 + \vec{\sigma}_2)|A\rangle = 0$$

$$\Rightarrow \langle A|(\vec{\sigma}_1 \cdot \hat{n}_1)(\vec{\sigma}_2 \cdot \hat{n}_2)|A\rangle = -\langle A|(\vec{\sigma}_1 \cdot \hat{n}_1)(\vec{\sigma}_1 \cdot \hat{n}_2)|A\rangle.$$

We can write the expectation value (4. 2) as following

$$-\langle A|(\vec{\sigma}_1\cdot\hat{n}_1)(\vec{\sigma}_1\cdot\hat{n}_2)|A\rangle = -\frac{1}{\sqrt{2}}\langle A|(\vec{\sigma}_1\cdot\hat{n}_1)(\vec{\sigma}_1\cdot\hat{n}_2)(|(\hat{n}_2+),(\hat{n}_2-)\rangle - |(\hat{n}_2-),(\hat{n}_2+)\rangle). \tag{4. 3}$$

After operating $(\vec{\sigma}_1 \cdot \hat{n}_2)$ to the state, we can write it as follows. If $\hat{n}$ is an arbitrary unit vector with polar and azimuthal angles θ and ϕ, we have the following equalities

$$|\hat{n}+\rangle = \cos\frac{\theta}{2}e^{-i\phi/2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi/2}|-\rangle,$$

$$|\hat{n}-\rangle = -\sin\frac{\theta}{2}e^{-i\phi/2}|+\rangle + \cos\frac{\theta}{2}e^{i\phi/2}|-\rangle.$$

If we choose $\hat{n}_1$ as the polar axis and $\hat{n}_2$ as the polar angle θ with respect to it, we have

$$|\hat{n}_2+\rangle = \cos\frac{\theta}{2}|\hat{n}_1+\rangle + \sin\frac{\theta}{2}|\hat{n}_1-\rangle,$$

$$|\hat{n}_2-\rangle = -\sin\frac{\theta}{2}|\hat{n}_1+\rangle + \cos\frac{\theta}{2}|\hat{n}_1-\rangle.$$

After doing proper calculations we get

$$< (\vec{\sigma}_1 \cdot \hat{n}_1)(\vec{\sigma}_2 \cdot \hat{n}_2) >= -\cos\theta \tag{4. 4}$$

where θ is the angle between two unit directions.

Now, to see the contradiction calculate (4. 1). Since our state is singlet, if we measure both spins along the same direction we get perfect anti-correlation, $A_\lambda(\hat{n}) = -B_\lambda(\hat{n})$

$$E(\hat{n}_1, \hat{n}_2) = -\int A_\lambda(\hat{n}_1)A_\lambda(\hat{n}_2)\rho(\lambda)d\lambda. \tag{4. 5}$$

Here we should note that when $\hat{n}_1 = \hat{n}_2$ we get (4. 1) and (4. 2) are both equal to -1. Now, introducing another unit vector $\hat{n}_3$

$$\begin{aligned} E(\hat{n}_1,\hat{n}_2) - E(\hat{n}_1,\hat{n}_3) &= -\int (A_\lambda(\hat{n}_1)A_\lambda(\hat{n}_2) - A_\lambda(\hat{n}_1)A_\lambda(\hat{n}_3))\rho(\lambda)d\lambda \\ &= \int (-A_\lambda(\hat{n}_1)A_\lambda(\hat{n}_2))(1 - A_\lambda(\hat{n}_1)A_\lambda(\hat{n}_3))\rho(\lambda)d\lambda, \end{aligned} \quad (4.\ 6)$$

since $A_\lambda(\hat{n}_2)$ is equal to ± 1 its square is equal to 1. Absolute value of the first term in the integral is equal to 1. Second term in the integral is whether 0 or 2, its absolute value is equal to itself

$$|E(\hat{n}_1,\hat{n}_2) - E(\hat{n}_1,\hat{n}_3)| \leq \int (1 - A_\lambda(\hat{n}_1)A_\lambda(\hat{n}_3))\rho(\lambda)d\lambda = 1 + E(\hat{n}_2,\hat{n}_3), \quad (4.\ 7)$$

where we use normalization. We can rewrite the Bell's inequality:

$$|E(\hat{n}_1,\hat{n}_2) - E(\hat{n}_1,\hat{n}_3)| - E(\hat{n}_2,\hat{n}_3) \leq 1. \quad (4.\ 8)$$

Now suppose all three unit vectors $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$ lie in the xy plane and having azimuthal angles 0, $\pi/3$ and $2\pi/3$ respectively. Then we have

$$E(\hat{n}_1,\hat{n}_2) = E(\hat{n}_2,\hat{n}_3) = -\frac{1}{2}, E(\hat{n}_1,\hat{n}_3) = \frac{1}{2}.$$

This implies the violation of inequality (4. 8)

$$\frac{3}{2} \geq 1.$$

4.1.2 CHSH inequality

Clauser, Horne, Shimony, and Holt proposed an experimentally more realizable inequality [75]. They considered photon, instead of spin 1/2-particle. Photon can be treated as a qubit since it has two independent polarization states, although it is a spin-1 particle, and it has eigenvalues ± 1.

Linear polarization states are denoted as $|x\rangle$ (horizontal) and $|y\rangle$ (vertical). A polarization analyzer can be used to measure the linear polarization of a photon along any axis in the xy plane and a rotation on this plane is given as following

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$x(\theta)$ and $y(\theta)$ are linear polarization eigenstates in the rotated frame. On this rotated frame we can define an operator (analog of $\vec{\sigma}\cdot\hat{n} = |\hat{n}+\rangle\langle\hat{n}+| - |\hat{n}-\rangle\langle\hat{n}-|$)

$$\tau(\theta) = |x(\theta)\rangle\langle x(\theta)| - |y(\theta)\rangle\langle y(\theta)|. \tag{4. 9}$$

Expectation value of product outcomes can be written as

$$\langle \tau(\theta_1)^A \tau(\theta_2)^B \rangle. \tag{4. 10}$$

Suppose $A_\lambda(\hat{n}) = \pm 1$ and $B_\lambda(\hat{n}) = \pm 1$ are the results of measurements of first and second photons along the specified directions. We have the following equalities

$$(A_\lambda(\hat{n}_1))^2 (A_\lambda(\hat{n}_1^{'}))^2 (B_\lambda(\hat{n}_2))^2 (B_\lambda(\hat{n}_2^{'}))^2 = 1$$

and

$$(A_\lambda(\hat{n}_1)B_\lambda(\hat{n}_1))(A_\lambda(\hat{n}_1)B_\lambda(\hat{n}_1^1))(A_\lambda(\hat{n}_1^{'})B_\lambda(\hat{n}_1))(-A_\lambda(\hat{n}_1^1)B_\lambda(\hat{n}_1^{'})) = -1.$$

Among above five terms, whether one of them is negative or three of them are negative. So we can write the following inequality

$$|A_\lambda(\hat{n}_1)B_\lambda(\hat{n}_1) + A_\lambda(\hat{n}_1)B_\lambda(\hat{n}_1^{'}) + A_\lambda(\hat{n}_1^{'})B_\lambda(\hat{n}_1) - A_\lambda(\hat{n}_1^{'})B_\lambda(\hat{n}_1^{'})| \leq 2. \tag{4. 11}$$

To see the contradiction, let us use the state $|\phi^+\rangle = (|xx\rangle + |yy\rangle)/\sqrt{2}$

$$\begin{aligned}\langle\phi^+|\tau(\theta_1)^A\tau(\theta_2)^B|\phi^+\rangle &= \langle\phi^+|\tau(0)^A\tau(\theta_2-\theta_1)^B|\phi^+\rangle \\ &= \langle\phi^+|\tau(\theta_2-\theta_1)^B\frac{1}{\sqrt{2}}(|xx\rangle - |yy\rangle).\end{aligned} \tag{4. 12}$$

If we write the state $|\phi^+\rangle$ explicitly in the above equation, and take the partial trace of it, we would get the following equality in the basis of second photon

$$\begin{aligned}\langle\phi^+|\tau(\theta_1)^A\tau(\theta_2)^B|\phi^+\rangle &= \frac{1}{2}(\langle x|\tau(\theta_2-\theta_1)^B|x\rangle - \langle y|\tau(\theta_2-\theta_1)^B|y\rangle) \\ &= \cos^2(\theta_2-\theta_1) - \sin^2(\theta_2-\theta_1) = \cos[2(\theta_2-\theta_1)].\end{aligned} \tag{4. 13}$$

Consider the case, where each photon has an angle $\pi/4$ to another, if we impose this condition to (4. 11) using (4. 13) we see the violation

$$2\sqrt{2} \geq 2. \tag{4. 14}$$

Note that, we can see the violation of CHSH inequality (4. 11) by using various states. In this case we should change the orientation of photons. We could also see the violation in a different way, which is similar to the way we use in our pentagram inequality. Denote the operators as $\hat{A}(\hat{n})$ and $\hat{B}(\hat{n})$ which have corresponding eigenvalues in the inequality (4. 11) (these operators are Pauli operators). If we assume all four operators $\hat{A}(\hat{n}_1)$, $\hat{A}(\hat{n}_1^{'})$, $\hat{B}(\hat{n}_1)$, and $\hat{B}(\hat{n}_1^{'})$ have a hidden joint probability distribution, we could take the average of (4. 11) and could have the following inequality

$$|\langle \hat{A}_\lambda(\hat{n}_1)\hat{B}_\lambda(\hat{n}_1)\rangle + \langle \hat{A}_\lambda(\hat{n}_1)\hat{B}_\lambda(\hat{n}_1^{'})\rangle + \langle \hat{A}_\lambda(\hat{n}_1^{'})\hat{B}_\lambda(\hat{n}_1)\rangle - \langle \hat{A}_\lambda(\hat{n}_1^{'})\hat{B}_\lambda(\hat{n}_1^{'})\rangle| \leq 2. \tag{4. 15}$$

We can directly see the violation from this equation, by using proper directions and the states.

4.2 Pentagram inequality

We consider a single spin-1 particle as an example [86]. Its Hilbert space $\mathcal{H} = R^3 \otimes C$ can be conveniently identified with complexification of three-dimensional Euclidian space R^3. The spin group SU(2), locally isomorphic to SO(3), acts on $\mathcal{H}$ by rotations of R^3. The Hilbert space inherits from R^3 bilinear cross products, that allows to express spin projection operator onto direction ℓ,

$$S_\ell \psi = i\ell \times \psi = |+1\rangle\langle +1| - |-1\rangle\langle -1|$$

. It has three eigenstates, one real $|0\rangle = \ell$ and two complex conjugate $|\pm 1\rangle = (m \pm in)/\sqrt{2}$, where $\{\ell, m, n\}$ is as an orthonormal basis in R^3. The latter two states are coherent [6, 7, 39], but here we are primarily concerned with the *entangled* spin state $|0\rangle$.

As we've mentioned above, Bell's constraints are just compatibility conditions for partial joint distributions of *commuting* observables. So, to create a Bell type condition we need commuting observables. Such observables in spin-1 system can be constructed as follows. Let's define *reflection operator* in plane orthogonal to $\ell \in R^3$

$$R_\ell = I - 2|\ell\rangle\langle\ell| = 2S_\ell^2 - I, \tag{4. 16}$$

where I denotes identity operator.For orthogonal directions $\ell \perp \ell'$ the operators commute $[R_\ell, R_{\ell'}] = 0$. Besides commutating, this operator has another advantage; it has two eigenvalues $r_\ell = \pm 1$.

Consider now a cyclic quintuplet of unit vectors $\ell_k \in R^3$ such that $\ell_k \perp \ell_{k+1}$ with indices taken modulo 5. Hereafter we call it a *pentagram*. Geometry of a regular pentagram is shown in figure 4. 1 For simplicity, with every pentagram we associate observable $R_k := R_{\ell_k}$ assuming value $r_k = \pm 1$ such that $[R_k, R_{k+1}] = 0$.

Note that for any $r_k = \pm 1$, we have the following equality

$$(r_1)^2(r_2)^2(r_3)^2(r_4)^2(r_5)^2 = (r_1r_2)(r_2r_3)(r_3r_4)(r_4r_5)(r_5r_1) = 1.$$

Among these five terms at least one of them should be $+1$, so the the following inequality holds

$$r_1r_2 + r_2r_3 + r_3r_4 + r_4r_5 + r_5r_1 \geq -3. \tag{4. 17}$$

Suppose now that the random quantities r_k, associated with observables R_k in state ψ, have a joint probability distribution. Then taking average of (4. 17) we get the following *pentagram inequality*

$$\langle R_1R_2\rangle + \langle R_2R_3\rangle + \langle R_3R_4\rangle + \langle R_4R_5\rangle + \langle R_5R_1\rangle \geq -3. \tag{4. 18}$$

It imposes a constraint on correlations between measurements of R_k caused by assumption of classical realism. The construction resembles that of the Clauser-Horne-Shimony-Holt inequality [75].

The pentagram inequality can be written in more transparent form using equations

$$R_kR_{k+1} = I - 2(|\ell_k\rangle\langle\ell_k| + |\ell_{k+1}\rangle\langle\ell_{k+1}|) \tag{4. 19}$$

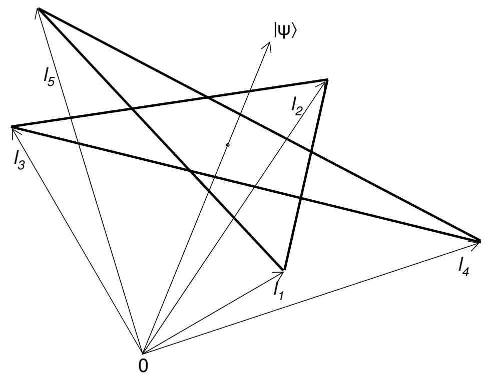

Figure 4. 1: Regular pentagram defined by cyclic quintuplet of unit vectors $\ell_i \perp \ell_{i+1}$. State vector $|\psi\rangle$ is directed along the symmetry axis of the pentagram.

that follow from (4. 16) and orthogonality $\langle \ell_k | \ell_{k+1} \rangle = 0$. This leads to the following version of the pentagram inequality

$$\sum_{kmod5} |\langle \ell_k | \psi \rangle|^2 \leq 2. \tag{4. 20}$$

Note that, $|\langle \ell_k | \psi \rangle|$ is just the angle between the unit vector ℓ_k and ψ. Let's test it for spin state $S_z = 0$ represented by unit vector ψ along z-axis, and a *regular pentagram* with 5-fold symmetry around z-axis. A simple calculation shows that in this case each ℓ_k has the same angle with z-axis, $|\langle \ell_k | \psi \rangle|^2 = \cos^2 \widehat{\ell_k z} = \frac{\cos(\pi/5)}{1+\cos(\pi/5)} = \frac{1}{\sqrt{5}}$, that violates the pentagram inequality

$$\sum_{kmod5} |\langle \ell_k | \psi \rangle|^2 = \sqrt{5} \approx 2.236 > 2.$$

Therefore the state $S_z = 0$ is nonclassical.

In biphoton setting, discussed below, the quantity $|\langle \ell | \psi \rangle|^2$ is just a coincidence rate in Hanbury Brown–Twiss interferometer. Due to 5-fold symmetry of the configuration only one such quantity should be actually measured to refute the hidden variables model.

In the inequality (4. 18) we can also return to original spin projection operators. To do this we should note that each operator product in (4. 18) is nothing but the rotation by angle π in the plane constituted by $\vec{\ell_i}$ and $\vec{\ell}_{i+1}$ i.e.

$$R_{\ell_i} R_{\ell_{i+1}} = 1 - 2S^2_{\ell_i \times \ell_{i+1}}. \tag{4. 21}$$

After observing the fact that $\vec{\ell_i}, \vec{\ell}_{i+1}and(\vec{\ell_i} \times \vec{\ell}_{i+1})$ are orthogonal to each others and span the whole space, we get $S^2_{\ell_i} + S^2_{\ell_{i+1}} + S^2_{\ell_i \times \ell_{i+1}} = s(s+1) = 2$. Thus, we can write the pentagonal inequality (4. 18) in terms of original spin projection operators as follows

$$\langle S^2_{\ell_1} \rangle_\psi + \langle S^2_{\ell_2} \rangle_\psi + \langle S^2_{\ell_3} \rangle_\psi + \langle S^2_{\ell_4} \rangle_\psi + \langle S^2_{\ell_5} \rangle_\psi \geq 3. \tag{4. 22}$$

This inequality can be tested experimentally by measuring S and calculating the average of S^2.

Finally, we can extend the pentagram inequality from spin 1 systems to two qubits formed by two spin-1/2 particle. The latter splits into skew symmetric spin-0 singlet and symmetric spin-1 triplet. We extend the reflection R_ℓ and spin S_ℓ operators into spin zero sector as identity and zero respectively. $R_\ell = 1 - 2|\ell\rangle\langle\ell|$ equality is still valid, R_ℓ and S_ℓ have same eigenvalues.

With this understanding, inequality (4. 17) and the pentagram inequality remain valid. However inequality (4. 22) should be modified, because relation with spin $R_k R_{k+1} = I - 2S^2_{\ell_k \times \ell_{k+1}}$ became more complicate due to spin, we can not use casimir operator here

$$\sum_{k mod 5} \langle \psi | S^2_{\ell_k \times \ell_{k+1}} | \psi \rangle \leq 4. \tag{4. 23}$$

By substitution $S_\ell = S^A_\ell + S^B_\ell$ we can write it as Bell condition for two qubit (squares of spin projection operators are equal to identity divided by two,for spin-1/2)

$$\sum_{k mod 5} \langle \psi | S^A_k S^B_k | \psi \rangle \leq 3, \tag{4. 24}$$

where we use shortcut $S^A_k := 2S^A_{\ell_k \times \ell_{k+1}}$ to reduce spin projections of the components to values ± 1. Since this inequality is based on measurements of spin, it couldn't detect entanglement in skew symmetric spin-0 sector.

This is the inequality shortly after decay, when the size of the two component system AB is still much smaller then aperture of an apparatus used for spin measurement. Inequality (4. 24) looks very similar to other Bell constraints [75], except for directions of spin projections at sites A and B that are always parallel. This allows to detect entanglement in closely tight systems, for which separate measurement of the components is unfeasible.

Note that by a unitary rotation one can transform every spin-1 state into *canonical form*

$$\psi = m\cos\varphi + in\sin\varphi,$$

where m, n are two fixed unit orthogonal vectors in R^3. Intrinsic properties of ψ are determined by the parameter $0 \le \varphi \le \frac{\pi}{4}$. For example, Wootters's concurrence $c(\psi)$ [22] of spin state ψ, considered as a symmetric state of two qubits, is equal to $\cos 2\varphi$ and coincides with a measure of entanglement for spin states introduced in [6]. The extremal values $c = 0$ and $c = 1$ correspond to coherent $S_z = 1$ and entangled $S_z = 0$ spin states respectively. The regular pentagram can detect nonclassical nature of the state ψ only for $c(\psi) > \frac{1}{\sqrt{5}}$. For states with smaller positive concurrence one has to use a skew pentagram containing two almost parallel vectors. [49].

As a physical example, we consider the case of *biphoton* [52]. The simplest three-photon interaction Hamiltonian has given before

$$H_{int} = g(b^+ a_H a_V + a_V^+ a_H^+ b) \tag{4. 25}$$

and the Stokes operators

$$\left\{ \begin{array}{lcl} S_x &=& \frac{1}{2}(a_H^+ a_V + a_V^+ a_H) \\ S_y &=& \frac{-i}{2}(a_H^+ a_V - a_V^+ a_H) \\ S_z &=& \frac{1}{2}(a_H^+ a_H - a_V^+ a_V) \end{array} \right. \tag{4. 26}$$

obey the spin commutation relations and act on the triplet states as spin-1 operators.

For the biphoton system the concurrence $c(\psi)$ is closely related with its *degree of polarization* $P(\psi) = \sqrt{1 - c(\psi)^2}$, that can be literally seen in classical polarization dependent intensity measurements [87, 88]. In contrast, the quantity $|\langle \ell|\psi\rangle|^2$

that enters into pentagram inequality (4. 20) requires a quantum measurement in Hanbury Brown–Twiss interferometer. Specifically it is equal to coincidence rate in the interferometer feeded by biphotons in state ψ while polarization filters inserted into its arms select photons in orthogonal polarization states given by opposite points $\pm\ell$ in Poincaré sphere. As we've seen above, to test the classical realism for neutrally-polarized state $|0\rangle$ and a regular pentagram we need only one experimental value $|\langle\ell|\psi\rangle|^2$ for $\cos^2\widehat{\ell\psi} = 1/\sqrt{5}$, that corresponds to the angle $\delta = \widehat{\ell\psi} \approx .8383$ radian. Quantum theory predicts $|\langle\ell|\psi\rangle|^2 = 1/\sqrt{5} \approx 0.4472$, while to refute hidden variable model we need $|\langle\ell|\psi\rangle|^2 > 0.4$. By some reason experimental data presented in [89, Fig. 8] fall far below of the theoretical curve $|\langle\ell|\psi\rangle|^2 = \cos^2\widehat{\ell\psi}$ in vicinity of the above value $\delta = .8383$ and provide no evidence for violation of the classical realism in biphoton system.

4.3 Summary

In conclusion, we have provided a first reliable test of the classical realism in local spin 1 system by construction of Bell's type inequality. A variation of this test can be applied to nonlocal systems as well, especially when they consist of very close components for which separate measurements are unfeasible. We have shortly discussed application of this approach to biphoton, where available experimental data are still insufficient for refuting hidden variables models.

Chapter 5

Conclusions

In this work, we have argued the existence of single spin-qutrit entanglement. The instructive significance of this system is that it allows twofold consideration as a single spin-1 object and as two qubits, defined in the symmetric sector of the Hilbert space. This correspondence allows us to interpret entanglement of single spin-qutrit as manifestation of quantum correlations between the intrinsic qubit degrees of freedom. We have shown that entanglement of single spin-qutrit particle may take place independent of whether or not the intrinsic qubits are separated. Thus, the single spin-qutrit entanglement does not fit conventional requirements of nonseparability and nonlocality. At the same time, the single spin-qutrit entanglement has all physical features of two-qubit entanglement. In particular, entangled states of a single spin-qutrit are squeezed and unentangled states are coherent like in the case of bipartite systems, and entangled states violates Bell's type condition. The latter is very important, it shows violation of classical realism without nonlocality.

In our analysis, we have used a general approach to quantum entanglement [30], which assigns the primary importance to the dynamic symmetry properties of physical systems. We have discussed a number of physical objects that can be prepared in entangled spin-qutrit states.

The obtained results show distinctly that the physical definition of entanglement [30], based on definition of basic observables and their quantum fluctuations, is more general than the previous definitions that appeal to nonlocality and nonseparability. As shown, using true and spin qutrits as illustrative examples, the presetting of basic observables plays a crucial role in the description of entanglement. In particular, it defines specific relativity of entanglement with respect to dynamic symmetry of physical system. The definition in terms of the variational principle (2. 25) can be used for investigation of entanglement of different physical objects, including elementary particle, quasi-particle excitations in condensed matter and so on. Thus, it essentially broaden the applicability of this notion beyond the bounds of quantum information. It is possible to say that the association of entanglement with quantum uncertainties of basic observables makes this notion to be ubiquitous in physics.

The consideration of entanglement beyond the conditions of nonlocality and nonseparability seems to be of high importance for the extraction of the physical nature of it. Relation of entanglement with manifestation of quantum uncertainties of basic observables at their extreme can lead to a new interpretation of a number of physical phenomena. The physical condition of complete entanglement as extreme of quantum fluctuations can be important for understanding of low stability of entangled states of particles. The association of the low stability π^0 meson with the maximal order of quantum fluctuations provides an example.

The obtained result about the single-particle entanglement for the spin-qutrit system is clearly valid for all systems with high enough dynamic symmetry $\mathrm{SU}(N)$ at $N \geq 3$.

The possibility of experimental observation of single-particle entangled states represents a problem of high importance and deserves special discussion. Let us only remind that, we can experimentally observe violation of Bell type condition by a single particle entangled state, this state is nonclassical. Also the decay of a single entangled SU(2) qutrit into two entangled qubits may be used for this aim.

We can not measure entanglement of mixed states with variance . In fact, measurement of entanglement of mixed states is known only for two qubits [23]). In the particular case of a single SU(2) qutrit, the mixed state entanglement can be quantified in the same way as the two-qubit entanglement.

Appendix A

Algebra of observables

A *group* is a class G of objects $g_1, g_2, ...$, on which a multiplication operation $\odot$ is defined with the following properties [90]:

- $\forall\ x, y \in G \Rightarrow x \odot y \in G$;
- G contains an identity element e, $e \odot x = x \odot e = x$;
- For $\forall\ x \in G$ there is an inverse element x^{-1}, $x^{-1} \odot x = x \odot x^{-1} = e$;
- Associativity holds for $\forall\ x, y$, and $z \in G$, $(x \odot y) \odot z = x \odot (y \odot z)$.

Multiplication operation can be any operation, e.g. it is additivity for the group of integers.

Transformation group in a quantum mechanical system is represented by a set of unitary operators. Representation of such a group should preserve the multiplication law

$$D(x)D(y) = D(x \odot y)$$

.

A group of unitary operators, in which the group elements are labelled by a set of continuous parameters, is called *compact Lie group*. Any group element of a Lie group can be represented as

$$D(X_n) = \exp(i\alpha_n X_n),$$

continuous parameters α_n and X_n $(n = 1, \dots N)$ are linearly independent Hermitian operators. X_n are the basis of the vector space composed of all linear combinations of $\alpha_n X_n$, and they are called generators of the group. The dimension of the representation is the dimension of the vector space that it acts.

The generators and the commutation relations define the *Lie algebra* and associated Lie group (A Lie algebra is a logarithm of a Lie group, and a Lie group is an exponential of a Lie algebra).

To find the properties of Lie algebra from commutation relations consider the *product*

$$\exp(i\alpha X_b)\exp(i\alpha X_a)\exp(-i\alpha X_b)\exp(-i\alpha X_a) = 1 + \alpha^2[X_a, X_b] + ...$$

product of the group element should also be group element, i.e. $[X_a, X_b] = i f_{abc} X_c$ where f_{abc} are called *structure constants*. We can extract some properties of the generators of the Lie algebra from structure constants

- $[X, X] = 0$;
- $[X_a + X_b, X_c] = [X_a, X_c] + [X_b, X_c]$;
- $[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$.

Simplest noncommutative Lie algebra is represented by three generators $J_a, a = 1, 2, 3$ with $f_{abc} = \varepsilon_{abc}$. This is the angular momentum algebra, and generators are the rotations in the rotation group. This group is called $SU(2)$, where S stands for special, and U stands for unitary.

A group of $N \times N$ matrices, which is the group of the rotations in an N-dimensional real vector space is called $SO(N)$, where S stands for special and O stands for orthogonal.

We can characterize rotations using both $SU(2)$ and $SO(3)$, with 2 to 1 correspondence, they are locally isomorphic.

Appendix B

Density matrix

A quantum state of a system represents the complete description of its physical properties. But sometimes we may not know the state precisely. A *density matrix* (density operator) is used to describe the statistical state of quantum system. When a quantum mechanical system undergoes general quantum operation (e.g. measurement) we need density matrix. The systems that we need their density matrices maybe systems in thermal equilibrium, entangled two subsystems, etc.

If a state is not reducible to a convex combination of other statistical states i.e. can be represented by a single state, the system is said to be in a pure state, otherwise system is in a mixed state. For pure states, density matrix is given by projection operator of this state $\rho = |\psi\rangle\langle\psi|$. For a mixed state, where the system is in the quantum mechanical state $|\psi_i\rangle$ with probability p_i, the density matrix is sum of projectors weighted with corresponding probabilities

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

where $p_i > 0$, $\sum_i p_i = 1$ (convex combination).

This decomposition is not unique and all convex decompositions are physically equivalent. Expectation value of an operator is given as

$$\langle\hat{A}\rangle = Tr(A\rho),$$

for pure states it reduces to $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$. Some properties of density matrix can be given as following

- ρ is hermitian, $\rho = \rho^\dagger$;
- ρ is positive, $spec(\rho) \geq 0$;
- $Tr\rho = 1$, $Tr\rho^2 \leq 1$, equality holds for pure states.

B.1 Reduced density matrix

Hilbert space $\mathcal{H}$ of a $N-$partite system is associated with the tensor product subspaces

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N.$$

As an example of tensor product operation, consider the two two-level systems

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

equivalently

$$|\phi_1\rangle \otimes |\phi_2\rangle = |\phi_1, \phi_2\rangle.$$

A pure state is separable on $N-$dimensional $\mathcal{H}$ if it can be written as a direct product of n states each belonging to different subsystems. A mixed state is separable, if it can be written as convex sum of product of n states. If the states are separable, they are unentangled.

A mixed state called $\nu-$separable if it can be written written as $(\nu \leq N)$

$$\rho = \sum_i p_i \rho_i^1 \otimes ... \otimes \rho_i^\nu$$

where superscripts denotes the subspace.

Any pure state on $N-$dimensional $\mathcal{H}$ can be written as

$$|\Psi\rangle_{n_1,n_2,...,n_N} = \sum_{n_1,n_2,...,n_N} c_{n_1,n_2,...,n_N}|u^{(1)}_{n_1}\rangle \otimes |u^{(2)}_{n_2}\rangle \otimes ...|u^{(N)}_{n_N}\rangle \qquad (2.\ 1)$$

where $|u^{(i)}\rangle$ denotes the complete basis of the Hilbert space of the subspace $\mathcal{H}_i$, with $n_i = 1, ..., \dim \mathcal{H}_i$. Due to normalization condition $\sum_{n_1,n_2,...,n_N} |c_{n_1,n_2,...,n_N}|^2 = 1$.

Reduced density matrix is obtained by taking the partial trace of the whole system, to get information about the subsystem that we interested. Followings are the examples:

$$\rho_1 = \sum_{n_2,n_3,...,n_N} \langle n_2, n_3, ..., n_N|\rho_{1,2,...,N}|n_2, n_3, ..., n_N\rangle$$

or

$$\rho_2 = \sum_{n_1,n_3,...,n_N} \langle n_1, n_3, ..., n_N|\rho_{1,2,...,N}|n_1, n_3, ..., n_N\rangle.$$

As an example consider bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ where $|a_i\rangle$ and $|b_i\rangle$ are the basis of the corresponding subspaces. Pure state (2. 1) becomes

$$|\Psi\rangle_{A,B} = \sum_{i,j} c_{i,j}|a_i\rangle \otimes |b_j\rangle = |a_i, b_j\rangle$$

and density matrix is equal to $\rho_{A,B} = |\Psi\rangle_{A,B\,A,B}\langle\Psi|$. Consider the reduced density matrix

$$\begin{aligned}\rho_A &= Tr_B(|\Psi\rangle_{A,B\,A,B}\langle\Psi|) \\ &= \sum_{i,i',j} c^*_{i,j}c_{i',j}|a_i\rangle\langle a_{i'}|.\end{aligned} \qquad (2.\ 2)$$

Consider the case that we only want to know the expectation value of an observable $\hat{O}_A$ acting on subsystem A

$$\begin{aligned}\langle \hat{O}_A \otimes I_B\rangle &=_{A,B} \langle\Psi|\hat{O}_A \otimes I_B|\Psi\rangle_{A,B} \\ &= \sum_{i,i',j,j'} c^*_{i,j}c_{i',j'}\langle a_i, b_j|\hat{O}_A \otimes I_B|a_{i'}, b_{j'}\rangle \\ &= \sum_{i,i',j} c^*_{i,j}c_{i',j}\langle a_i|\hat{O}_A|a_{i'}\rangle \\ &= Tr(\hat{O}_A\rho_A)\end{aligned} \qquad (2.\ 3)$$

ρ_A is enough to get information, we do not need the density matrix of the all system.

Appendix C

Qubit

Qubit is the abbreviation of the *"quantum bit"*. As bit is indivisible unit of classical information, qubit is corresponding unit of quantum information. Qubit is a two level system, this is the smallest nontrivial Hilbert space (an $n-$level quantum system is called *qunit*).

Qubit is represented by Hilbert space $\mathcal{H}_2$ with basis $|0\rangle, |1\rangle$, algebraically equivalent to spin$-1/2$ system. A bit can take value either 0 or 1, but a qubit state is a linear superposition of two states $|0\rangle$ and $|1\rangle$, it can be both with some probability.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, |\alpha|^2 + |\beta|^2 = 1.$$

Observables of two level system are given by Pauili operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They form an infinitesimal representation of the $s\ell(2, C)$ algebra. There are several physical realizations of a qubit:

- *Spin*$-1/2$ *system*

  In this case $\{|0\rangle, |1\rangle\}$ basis can be interpreted as spin-up and spin-down $\{|\uparrow\rangle, |\downarrow\rangle\}$ states.

- *Two level atom*

  Interaction of a two-level atom with EM field is mathematically equivalent to spin$-1/2$ particle. In this case basis are, ground and excited levels of an atom $\{|g\rangle, |e\rangle\}$. Atoms have many levels, we can select two of them by considering conservation of energy and selection rules (conservation of parity, angular momentum).

- *Polarization states of a photon*

  Although photon is a spin-1 particle, it has only two linearly independent polarization states horizontal and vertical (or left and right circular) polarizations.

A qubit can be geometrically represented by Bloch sphere (Poincaré sphere in the case of polarization states). A general qubit state can be written as follows

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle.$$

Density matrix of this state is equal to

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}(I + \hat{P}\cdot\vec{\sigma}),$$

where the unit vector $\hat{P} = (\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta)$ specifies a unique point on the unit sphere of Euclidian space R^3.

For mixed states there maybe a decomposition

$$\rho = \lambda\rho(\hat{P}_1) + (1-\lambda)\rho(\hat{P}_2).$$

This physically means that, the vector representing this state on Bloch sphere is no more a unit vector

$$\rho = \rho(\vec{P}') \Rightarrow |\vec{P}'| = |\lambda\hat{P}_1 + (1-\lambda)\hat{P}_2| \leq 1.$$

Pure states are represented on the surface of the Bloch sphere, but mixed states are represented just inside of the sphere.

Components P_i of the vector $\vec{P} = (P_x, P_y, P_z)$ can be written as $P_i = f_i^{(+)} - f_i^{(-)}$, where $f_i^{(\pm)}$ is the fraction of particles that are at the eigenstates $|\pm\rangle$ of

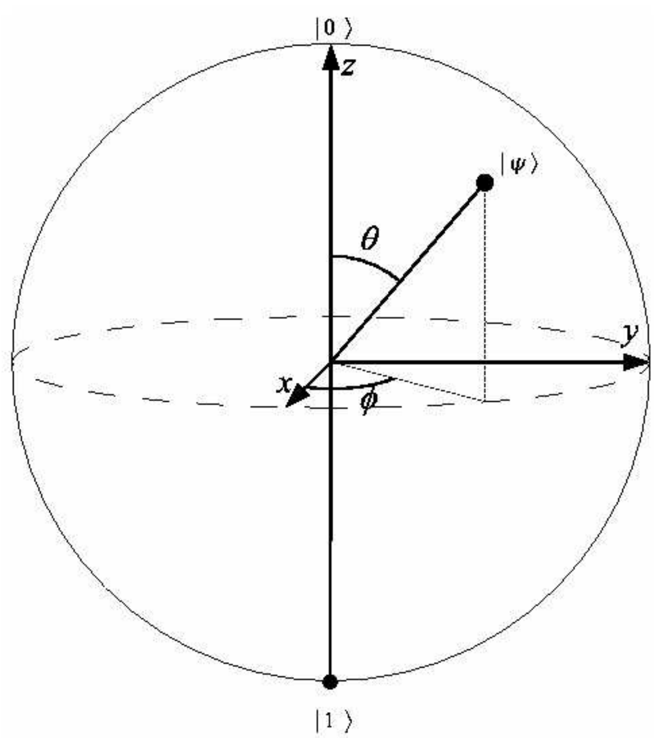

Figure 3. 1: Bloch sphere. $|0\rangle$ and $|1\rangle$ states are represented by the points $(0, 0, 1)$ and $(0, 0, -1)$ on Bloch sphere respectively.

σ_i. P_i gives us the fraction aligned in the direction $\hat{e}_i$, and called polarization. Magnitude of the polarization vector $P = |\vec{P}|$ has the range $0 \leq |P| \leq 1$, called *degree of polarization*. It is equal to 1 for pure states, pure states are unpolarized.

C.1 Symmetric two-qubit state

If we exclude the antisymmetric sector of two-qubit space, a general two-qubit state (2. 14) becomes:

$$|\psi\rangle = a|00\rangle + b(|01\rangle + |10\rangle) + d|11\rangle.$$

Using symmetric triplet basis, we can write its representation in spin-1 system as follows

$$|\psi\rangle = a|1\rangle + \sqrt{2}b|0\rangle + d|-1\rangle = a|1\rangle + b'|0\rangle + d|-1\rangle.$$

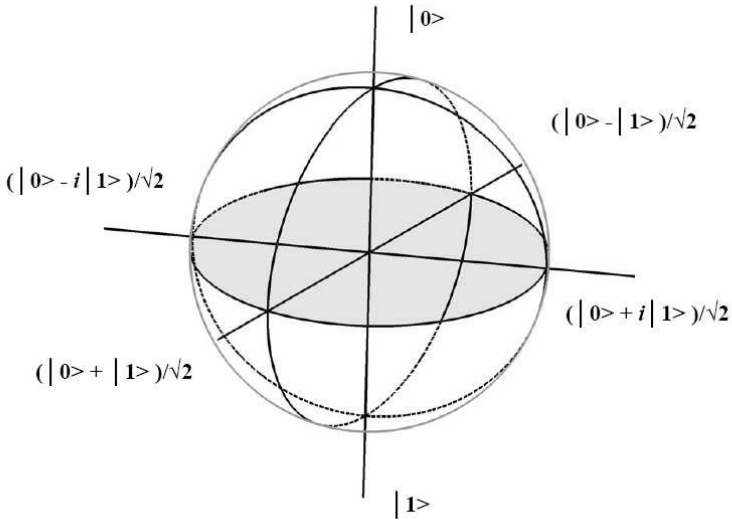

Figure 3. 2: $|0\rangle$ ($|1\rangle$) state, represents spin up (down) $|\uparrow\rangle$ ($|\downarrow\rangle$) state for spin$-1/2$ particle; ground (excited) level $|e\rangle$ ($|g\rangle$) for two-level atom, and right (left) circularly polarized state $|R\rangle(|L\rangle)$ for polarization of a photon.

Considering concurrence for two-qubit (2. 15), we can write the concurrence for single qutrit state as follows

$$C = 2|ad - b'^2/2|.$$

This measure exactly coincides with the measure (2. 29) that we get from variance.

An amount of entanglement carried by a mixed single spin qutrit state can be calculated in the same way as for two qubits through the use of Wootters' concurrence [23]. Namely

$$\mu(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3), \tag{3. 1}$$

where λ_i are the square roots of the eigenvalues of the (3×3) matrix $\rho F \rho^* F$, in decreasing order. Here the "spin-flip" transformation matrix F is defined for the spin qutrit state as follows

$$F = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

It corresponds to the Wootters' spin-flip transformation $\sigma_y \otimes \sigma_y$ defined in the symmetric sector of the four-dimensional Hilbert space.

For example, the concurrence (3. 1) of the "symmetric" Werner state

$$\rho_W = \frac{x}{3}(|+1\rangle\langle+1| + |0\rangle\langle 0| + |-1\rangle\langle -1|) + (1-x)|0\rangle\langle 0|,$$

which represents superposition of completely mixed and completely entangled states, has the form

$$C(\rho_W) = \begin{cases} (1-4x/3), & \text{at } 0 \le x < 3/4 \\ 0, & \text{at } 3/4 \le x \le 1 \end{cases}$$

Remind that in the case of conventional two-qubit Werner state [91] the concurrence has the form

$$C(\rho_{W_{4d}}) = \begin{cases} (1-3x/2), & \text{at } 0 \le x < 2/3 \\ 0, & \text{at } 2/3 \le x \le 1 \end{cases}$$

Thus, entanglement of "symmetric" Werner state survives at higher admixture of completely chaotic state than that of Werner state in the whole space $\mathcal{H}_2 \otimes \mathcal{H}_2$.

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